There are no interesting modular forms of level $\Gamma_0(16)$

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Abstract

In this note I study newforms with reasonably large level at 2. The two levels that are of interest are level $\Gamma_0(16N)$ and $\Gamma_0(64N)$. There is a natural source of such newforms, namely twisting newforms of lower level by quadratic characters of conductor 4 or 8. Some simple calculations suggest that this construction exhausts all of the newforms. These calculations are justified by coverting this problem into a Galois-theoretic statement about $G_{\mathbb{Q}_2}$ and then proving said statement.

1 Introduction

This observation started with the following calculation (numbers are approximations):

Group	$\Gamma_0(p)$	$\Gamma_0(2p)$	$\Gamma_0(4p)$	$\Gamma_0(8p)$	$\Gamma_0(16p)$
$\dim(\mathcal{S}_2(\Gamma))$	$\frac{p}{12}$	$\frac{p}{4}$	$\frac{p}{2}$	p	$2p$
$\dim(\mathcal{S}_2(\Gamma)^{new})$	$\frac{p}{12}$	$\frac{p}{12}$	$\frac{p}{12}$	$\frac{p}{4}$	$\frac{p}{2}$

The first row of this table was computed by the following string of approximate equalities:

$$\dim(\mathcal{S}_2(\Gamma_0(N))) = g(X_0(N)) \approx -\frac{1}{2}\chi(X_0(N)) \approx \frac{1}{12}\deg(X_0(N) \to X(1)).$$

As for the second row, that was computed by noticing that there are $b-a+1$ ways to turn a newform for $\Gamma_0(2^a p)$ into a newform for $\Gamma_0(2^b p)$; namely $f(\tau) \to f(\tau)$, $f(\tau) \to f(2\tau)$, $\ldots$, $f(\tau) \to f(2^{b-a}\tau)$. This allows the second column to be computed from the first column.

The key observation to make is that $\frac{p}{12} + \frac{p}{12} + \frac{p}{12} + \frac{p}{4} = \frac{p}{2}$. There is a very natural reason to expect that $\frac{p}{12} + \frac{p}{12} + \frac{p}{12} + \frac{p}{4} \leq \frac{p}{2}$: if one lets χ_4 be the Dirchlet character of conductor 4 and $f = \sum a(n)q^n$ be a newform of level dividing $8p$. Then $g = \sum a(n)\chi_4(n)q^n$ is a newform of level exactly $16p$, attaining the promised inequality. However, the fact that there is (roughly) an equality is striking: it suggests that you can always "simplify" a newform of level 16 by twisting by χ_4.

One can turn the rough equality into an exact equality as follows: the terms in the table are the leading term in computing the number of forms. The errors are constants that depend only on what p is mod 12, and so if one gets an equality for 5, 7, 11, and 13 (and 3), then the following statement is true:

Claim 1.1. $\dim(\mathcal{S}_2(\Gamma_0(p))^{new})+\dim(\mathcal{S}_2(\Gamma_0(2p))^{new})+\dim(\mathcal{S}_2(\Gamma_0(4p))^{new})+\dim(\mathcal{S}_2(\Gamma_0(8p))^{new}) = \dim(\mathcal{S}_2(\Gamma_0(16p))^{new})$.

It is possible to check this by hand or by using an online table. In either case, one gets these numbers:

p	$\mathcal{S}_2(\Gamma_0(p))^{new}$	$\mathcal{S}_2(\Gamma_0(2p))^{new}$	$\mathcal{S}_2(\Gamma_0(4p))^{new}$	$\mathcal{S}_2(\Gamma_0(8p))^{new}$	$\mathcal{S}_2(\Gamma_0(16p))^{new}$
3	0	0	0	1	1
5	0	0	1	1	2
7	0	1	0	2	3
11	1	0	1	3	5
13	0	2	1	3	6

This raises some natural questions. In no particular order,

1. What if we replace χ_4 with some other character ramified only at 2?
2. What if we replace 2 with some other prime?
3. What if we replace Γ_0 with Γ_1?
4. What if we replace p with N (N odd)?
5. What if we replace $\mathcal{S}_2$ with $\mathcal{S}_k$?

Some of these have quick answers. Replacing Γ_0 with Γ_1 is dead on arrivial: there are forms for $\mathcal{S}_2(\Gamma_1(4p))$ that are still of level $\Gamma_1(4p)$ after twisting by χ_4, and there are forms of level $\Gamma_1(8p)$ that have level $\Gamma_1(32p)$ after twisting by χ_4. Replacing 2 with ℓ also doesn't work as the dimensions grow too fast. There are $\frac{\ell}{12}$ newforms for $\Gamma_0(\ell)$ and $\frac{\ell^2}{12}$ newforms for $\Gamma_0(\ell^2)$ meaning not all of those (even with no level away from ℓ) can be twists of lower level forms.

Replacing weight 2 with weight k for some even value of k has a future though, as that just multiplies all the numbers on the first table by $k-1$. As for replacing χ_4 with another character, one can see that there are only two possibilities: the two characters of conductor 8. I will let χ_8 be the even character of conductor 8. Recreating the table above but going on until 64 and replacing weight 2 with weight k gives the following table:

Group	$\Gamma_0(p)$	$\Gamma_0(2p)$	$\Gamma_0(4p)$	$\Gamma_0(8p)$	$\Gamma_0(16p)$	$\Gamma_0(32p)$	$\Gamma_0(64p)$
$\dim(\mathcal{S}_k(\Gamma))$	$\frac{p(k-1)}{12}$	$\frac{p(k-1)}{4}$	$\frac{p(k-1)}{2}$	$p(k-1)$	$2p(k-1)$	$4p(k-1)$	$8p(k-1)$
$\dim(\mathcal{S}_k(\Gamma)^{new})$	$\frac{p(k-1)}{12}$	$\frac{p(k-1)}{12}$	$\frac{p(k-1)}{12}$	$\frac{p(k-1)}{4}$	$\frac{p(k-1)}{2}$	$p(k-1)$	$2p(k-1)$

Finally, this observation seems to be an observation about the nature of 2-dimensional representations of $G_{\mathbb{Q}_2}$ rather than an statement about the geometry of modular curves, and this suggests that replacing p with N is perfectly fine and will not change what is true. Precisely, I will prove the following theorem and then deduce the statement about modular froms from it:

Theorem 1.2. *Let $\rho : G_{\mathbb{Q}_2} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a continuous representation. Assume further that* $\det(\rho)$ *is unramified.*

- *If the conductor of ρ is 4, then the conductor of $\rho \otimes \chi_4$ is strictly less than 4.*
- *If the conductor of ρ is 6, then the conductor of $\rho \otimes \chi_8$ is strictly less than 6.*

Here, we are viewing χ_4 and χ_8 as characters of $G_{\mathbb{Q}_2}$, and are taking an additive choice of conductor. This theorem has as a corollary the precise version of the observations made above:

Corollary 1.3. *If $f = \sum a_f(n)q^n$ is a normalized newform in $\mathcal{S}_k(\Gamma_0(16N))$, then there is a newform $g = \sum a_g(n)q^n$ in $\mathcal{S}_k(\Gamma_0(2^aN))$ with $a < 4$ and $a_f(n) = \chi_4(n)a_g(n)$. Similarly, if f is a normalized newform in $\mathcal{S}_k(\Gamma_0(64N)$ then there is a normalized newform g in $\mathcal{S}_k(\Gamma_0(2^aN))$ with $a < 6$ and $a_f(n) = \chi_8(n)a_g(n)$.*

The awkwardness of the phrasing of this corollary is due to the fact that, e.g. in the first case, $a_f(n)\chi_4(n)$ is not necessarily equal to $a_g(n)$ if n is even. The corollary is immediate from the theorem: if f is a newform in $\mathcal{S}_k(\Gamma_0(16N))$, then there is a $\rho_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with conductor 4 at 2 and $\det(\rho) = \epsilon^{k-1}$ (ϵ is the cyclotomic character). One then has that $\rho_{f,\ell} \otimes \chi_4$ is modular as well, with strictly smaller conductor at 2 and the same conductor away from 2. Thus, $\rho_{f,\ell} \otimes \chi_4 = \rho_{g,\ell}$ for some modular form g, and then one can just unwind the statment about the fourier coefficents of f and g by computing the trace of $Frob_p$ on both sides of the equality $\rho_{f,\ell} \otimes \chi_4 = \rho_{g,\ell}$.

2 Proof

The proof will break up into cases. There are 3 possibilities for ρ:

- ρ is the sum of characters.
- ρ is an extension of characters.
- ρ is an irreducible representation.

For the first two items in the list, I will state what the conductor is for a general representation of that form, and enumerate all possible choices that lead to conductors less than or equal to 64. For the third item on the list, it is simpler to give an "all at once" argument.

If $\rho = \phi_1 \oplus \phi_2$, then the conductor of ρ is the product of the conductors of the ϕ_is. Since $\det(\rho)$ is unramified, one has that $\phi_1|I = (\phi_2|I)^{-1}$; in particular ϕ_1 and ϕ_2 have the same conductor. If the conductor of ϕ_1 is less than 16, then $(\phi_1|I)^{-1} = \phi_1|I$, which proves the claim for the sum of characters.

If ρ is an extension of ϕ_1 and ϕ_2, then the conductor of ρ is the product of the conductors of ϕ_1 and ϕ_2 unless both characters are unramified, in which case it is 2. The exact same argument as above applies.

To handle the final case, assume that ρ is an irreducible representation. Assume that ρ has conductor 16. Then $\rho^{I^{1,+}}$ is 2-dimensional and ρ^{I^1} is 0-dimensional: if there is an inertia subgroup I^x such that ρ^{I^x} is 1-dimensional, then ρ would be an extension of characters. Since ρ has conductor 16, the jump must be at I^2. Now, one has that $I^2/I^{2,+} \hookrightarrow \overline{\mathbb{F}_2}$. Write $\rho|_{I^1/I^{1,+}} = \chi_1 \oplus \chi_2$. Since one has that $det(\rho)|_{I^1} = \chi_1\chi_2$ must be trivial. Additionally, $I^1/I^{1,+}$ is an $\mathbb{F}_2$-vector space, so $\chi_1 = \chi_2$ Thus, one has that $\rho|_{I^2/I^{2,+}} = \chi \oplus \chi$ for some character χ. But χ must be stable under the action of $G_{\mathbb{Q}_2}/I$ and so there is only one possability: $\chi = \chi_4$. Thus, $(\rho \otimes \chi_4)^{I^1} = \rho$, which means that the conductor of $\rho \otimes \chi_4$ is less than 16.

The exact same argument as above applies to the case where the conductor of ρ is 64, where you replace I^1 with I^2 and χ_4 with χ_8.

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