Asymptotic Problems in Capillarity

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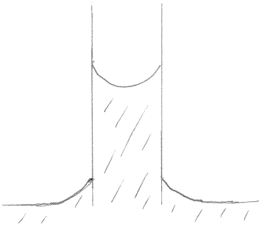
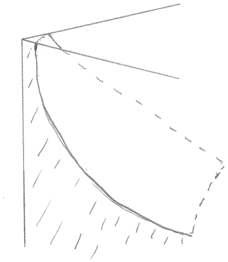
Two Types of Asymptotics



narrow tube



wedge

Capillary Surfaces

Minimize potential energy [Gauss 1830], [Finn 1986]

$$E[u] = \sigma \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx - \beta\sigma \int_{\partial\Omega} u\, ds + \frac{\rho g}{2} \int_\Omega u^2(x)\, dx$$

Height $u(x), x \in \Omega \subset \mathbb{R}^2, x = (x_1, x_2)$; surface tension/surface energy density σ; relative adhesion $\beta = \cos\gamma$; density of liquid minus density of air ρ; acceleration due to gravity g.
A variational argument gives the elliptic boundary value problem:

$$Nu = \kappa u \text{ in } \Omega$$

$$Tu \cdot \nu = \cos\gamma \text{ on } \partial\Omega$$

with $Nu = \nabla \cdot Tu$, $Tu = \frac{\nabla u}{\sqrt{1+|\nabla u(x)|^2}}$, ν is the exterior unit normal on $\partial\Omega, \kappa = \frac{\rho g}{\sigma}$.

Nu is twice the mean curvature of the surface $x_3 = u(x)$; γ is the angle between the surface $x_3 = u(x)$ and the cylinder $\partial\Omega \times \mathbb{R}$.

When there is a volume constraint the equation becomes $Nu = \kappa u + \lambda$.

Comparison Principle

[Concus and Finn 1974], [Finn and Hwang 1989]
Suppose $\kappa > 0$. If

$$Nu \geq \kappa u \text{ and } Nv \leq \kappa v \text{ in } \Omega$$

$$Tu \cdot \nu \leq Tv \cdot \nu \text{ or } u \leq v \text{ on } \partial\Omega \setminus Z, Z \subset \partial\Omega, \mathcal{H}^1(Z) = 0$$

then $u \leq v$ in Ω.

Note: Ω may be unbounded, no growth condition is required.

Narrow Circular Tube

Let Ω be a ball of radius a, $r = \sqrt{x_1^2 + x_2^2}$, $u = u(r)$. With non-dimensional variables $\overline{u} = \frac{u}{a}, \overline{r} = \frac{r}{a}$ and inclination angle $\overline{\psi}$, $\sin\overline{\psi} = \frac{\overline{u}_{\overline{r}}}{\sqrt{1+\overline{u}_{\overline{r}}^2}}$, $B = \kappa a^2$, we have the boundary value problem

$$(r \sin\overline{\psi})_r = Br\overline{u} \text{ for } 0 < \overline{r} < 1$$

$$\overline{\psi}(0) = 0 \text{ and } \overline{\psi}(1) = \frac{\pi}{2} - \gamma$$

Write u for $\overline{u}$ and ψ for $\overline{\psi}$ in what follows. Consider $0 \leq \gamma < \frac{\pi}{2}$. We seek an approximate solution for small B. [Siegel 2006]

The volume lifted is determined: $B\int_0^1 ru(r)\,dr = \cos\gamma$. Let u_0 be constant and lift this volume. Thus $u_0 = \frac{2\cos\gamma}{B}$. Define successive approximations $\{u_n(r)\}$ by

$$(r\sin\psi_{n+1})_r = Bru_n \text{ for } 0 < r < 1$$

$$\psi_{n+1}(0) = 0$$

$$B\int_0^1 ru_{n+1}\,dr = \cos\gamma$$

Finding u_{n+1} from u_n invloves two quadratures:

$$\sin\psi_{n+1}(r) = \frac{B}{r}\int_0^r su_n(s)\,ds$$

$$u_{n+1}(r) = u_{n+1}(0) + \int_0^r h_{n+1}(s)\,ds,\, h_{n+1} = \frac{\sin\psi_{n+1}}{\sqrt{1-\sin^2\psi_{n+1}}}$$

$$u_{n+1}(0) = u_0 - \int_0^1 (1 - r^2) h_{n+1}(r)\, dr$$

For $B \leq 6$ the approximations are defined, nonnegative and satisfy $\psi_n(1) = \frac{\pi}{2} - \gamma$. They also satisfy interleaving properties and error bound:

$$\psi_0 < \psi_2 < \cdots < \psi < \cdots < \psi_3 < \psi_1 \text{ for } 0 < r < 1$$

$$u_1(0) < u_3(0) < \cdots < u(0) < \cdots < u_2(0) < u_0$$

$$u_0 < u_2(1) < \cdots < u(1) < \cdots < u_3(1) < u_1(1)$$

$$|u - u_n| < \frac{1 - \sin\gamma}{\cos\gamma} B^n \leq B^n$$

The Comparison Principle gives $u > 0$ for $0 \leq r < 1$. The differential equation gives $\psi > 0 = \psi_0$ for $r > 0$. Since u and u_0 lift the same volume, they intersect once at r_0, $0 < r_0 < 1$: $u < u_0$ for $r < r_0$ and $u > u_0$ for $r > r_0$. As $(r \sin \psi_1)_r = Bru_0 > Bru = (r \sin \psi)_r$ for $0 < r < r_o$ and $\psi_1(0) = \psi(0) = 0$, $\psi_1 > \psi$ for $0 < r \leq r_0$. Similarly, $(r \sin \psi_1)_r < (r \sin \psi)_r$ for $r_0 < r < 1$ and $\psi_1(1) = \psi(1)$ implies $\psi_1 > \psi$ for $r_0 \leq r < 1$. Thus $\psi_1 > \psi$ for $0 < r < 1$. This then implies $u_1(0) < u(0)$ and $u_1(1) > u(1)$.

$$u_1(r) = \frac{2 \cos \gamma}{B} + \frac{\sin \gamma - \sqrt{1 - r^2 \cos^2 \gamma}}{\cos \gamma} + \frac{2}{3} \left(\frac{1 - \sin^3 \gamma}{\cos^3 \gamma} \right)$$

$$u(r; B) \sim \sum_{i=-1}^{\infty} h_i(r) B^i \text{ as } B \to 0$$

Wide Circular Tube

We seek an approximation for B large. The linearized equation $u_{rr} + \frac{u_r}{r} = Bu$ has solution $u_0(r) = A\mathcal{I}_0(\sqrt{B}r)$. For positive A, u_0 is a supersolution [Siegel 1980]:

$$Nu_0 = \frac{u_{0rr}}{(1+u_{0r}^2)^{\frac{3}{2}}} + \frac{u_{0r}}{r(1+u_{0r}^2)^{\frac{1}{2}}} \leq u_{0rr} + \frac{u_{0r}}{r} = Bu_0$$

The volume condition $B\int_0^1 ru_0(r)\,dr = \cos\gamma$ implies $A = \frac{\cos\gamma}{\sqrt{B}\mathcal{I}_0'(\sqrt{B})}$. By the Comparison Principle u_0 and u intersect once at r_0, $0 < r_0 < 1$: $u_0 > u$ for $0 \leq r < r_0$ and $u_0 < u$ for $r_0 < r \leq 1$. This leads to $\psi_1 > \psi$ for $0 < r < 1$. The succesive approximations are defined. A good error bound is not known.

Decay Estimate

Suppose Ω is unbounded. Let $Nu = \kappa u$ in Ω. For $x \in \Omega$, let $d = dist(x, \partial\Omega)$. Since the ball of raduis d about x is contained in Ω, we may compare u with the contact angle zero solution in the ball, leading to the decay estimate

$$|u(x)| < \frac{1}{\sqrt{\kappa}\mathcal{I}_0'(\sqrt{\kappa}d)}.$$

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Exterior to a Narrow Circular Tube

Let Ω be the region where $r > a$. Non-dimensionalize as before. We have the boundary value problem

$$Nu = Bu \text{ for } r > 1,\ \psi(1) = \gamma - \frac{\pi}{2},\ u \to 0 \text{ as } r \to \infty.$$

By the Decay Estimate the last condition is not needed. The solution to the linearized equation $u_0(r) = A\mathcal{K}_0(\sqrt{B}r)$ with A positive is again a supersolution.
Requiring $B\int_1^\infty ru_0(r)\,dr = \cos\gamma$ gives $A = -\frac{\cos\gamma}{\mathcal{K}_0'(\sqrt{B})}$.

$$\sin\psi_1(r) = -\frac{B}{r}\int_r^\infty su_0(s)\,ds = A\mathcal{K}_0'(\sqrt{B}r)$$

We have $\sin\psi_1(1) = -\cos\gamma$.

As in the interior problem u_0 intersects u once at r_0, $r_0 > 1$ and $\psi_1 < \psi$ for $r > 1$. This gives $u_1 > u$ for $r \geq 1$. Thus u_1 does not satisfy the volume condition.

From $u_0(1) < u(1) < u_1(1)$ and the asymptotics of the integral for $u_1(1)$:

$$u(1) = -\cos\gamma \ln\sqrt{B} + O(1) \text{ as } B \to 0.$$

Further successive approxiamtions are not defined.

[Miersemann 2006] establishes a full asymptotic series.

Corners

Let Ω contain a corner $|\theta| < \alpha$, $0 < r < r_0$. The solution is bounded near the origin when $\alpha + \gamma \geq \frac{\pi}{2}$ and unbounded near the origin when $\alpha + \gamma < \frac{\pi}{2}$. Consider the unbounded case.
[Concus and Finn 1970], [Miersemann 1993]
Let

$$h(\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{k\kappa}, \quad k = \frac{\sin\alpha}{\cos\gamma}$$

Then for some positive A, r_1, $r_1 \leq r_0$,

$$\left| u - \frac{h(\theta)}{r} \right| \leq Ar^3 \text{ for } 0 < r \leq r_1.$$

Tilted Wedge

Rotate a vertical wedge of half-angle α by angle τ about the x_2-axis. Let (x_1', x_2', x_3') be a coordinate system rotated the same way and $x_3' = u'(x_1', x_2')$ is the fluid interface.

$$Nu' = \kappa(\cos\tau\, u' + r'\cos\theta' \sin\tau) \text{ for } 0 < r' < r_0,\ |\theta'| < \alpha$$

$$Tu' \cdot \nu' = \cos\gamma \text{ for } \theta = \pm\alpha,\ 0 < r' < r_0$$

Let $h'(\theta') = \frac{h(\theta')}{\cos\tau}$. There exists positive $A,\ r_1 \leq r_0,\ \epsilon$ so that

$$\left| u' - \frac{h'(\theta')}{r'} \right| \leq Ar^{\epsilon} \text{ for } 0 < r' \leq r_1$$

Work in progress with Hanzhe Chen. Existence of a solution needs to be shown.

Cusps

Let Ω contain a cusp
$\Omega_0 = \{(x, y) : f_1(x) < y < f_2(x),\ 0 < x \leq x_0\}$ and

$$Nu = \kappa u \text{ in } \Omega_0$$

$$Tu \cdot \nu = \cos \gamma_1 \text{ for } y = f_1(x)$$

$$Tu \cdot \nu = \cos \gamma_2 \text{ for } y = f_2(x)$$

where $f_1(0) = f_2(0) = f_1'(0) = f_2'(0) = 0$ and $f_1(x) < f_2(x)$ for $0 < x \leq x_0$.
[Scholz 2004], [Aoki and Siegel 2012]

Case $\gamma_1 + \gamma_2 \neq \pi$

Change of variables: $s = x, \ t = \frac{2y-(f_1(x)+f_2(x))}{f_2(x)-f_1(x)}$. Then $0 < s \leq x_0, \ |t| < 1$.

Consider a three-term formal asymptotic expansion:

$$u(s,t) = \frac{\cos\gamma_1 + \cos\gamma_2}{f_2(s) - f_1(s)} + g(t)\frac{f_2'(s) - f_1'(s)}{f_2(s) - f_1(s)} + h(t)\frac{(f_2'(s) - f_1'(s))^2}{f_2(s) - f_1(s)}$$

Condition A: Take $h(t) = 0$,

$$f_2(s) - f_1(s) = o(f_2'(s) - f_1'(s)), \ \frac{f_2''(s) - f_1''(s)}{f_2(s) - f_1(s)} = o\left(\frac{f_2'(s) - f_1'(s)}{(f_2(s) - f_1(s))^2}\right),$$

$$\frac{f_2'''(s) - f_1'''(s)}{f_2'(s) - f_1'(s)} = o\left(\frac{1}{(f_2(s) - f_1(s))^2}\right) \text{ as } s \to 0^+$$

$$\Rightarrow g(t) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2} + C_1$$

Condition B:

$$f_2'(s) > f_1'(s) \text{ for } s > 0,\ f_2(s) - f_1(s) = o(f_2'(s) - f_1'(s)),$$

$$\frac{f_2''(s) - f_1''(s)}{f_2(s) - f_1(s)} = \alpha \frac{(f_2'(s) - f_1'(s))^2}{(f_2(s) - f_1(s))^2} + o\left(\frac{(f_2'(s) - f_1'(s))^2}{(f_2(s) - f_1(s))^2}\right),$$

$$\frac{f_2'''(s) - f_1'''(s)}{f_2'(s) - f_1'(s)} = O\left(\frac{(f_2'(s) - f_1'(s))^2}{(f_2(s) - f_1(s))^2}\right),$$

$$f_1'(s) + f_2'(s) = \delta(f_2'(s) - f_1'(s)) + o(f_2'(s) - f_1'(s))$$

$$f_1''(s) + f_2''(s) = O(f_2''(s) - f_1''(s)) \text{ as } s \to 0^+$$

$$\Rightarrow h(t) = -\frac{A}{4}\left(\delta t + \frac{t^2}{2}\right) + \frac{1-\alpha}{2A} g(t)^2 + C_2 \text{ and } C_1 = 0$$

where $A = \cos\gamma_1 + \cos\gamma_2,\ \alpha, \delta \in \mathbb{R}$.

Modifying the formal series gives supersolutions and subsolutions. Under Condition B, this leads to

$$u(s,t) = \frac{\cos\gamma_1 + \cos\gamma_2}{f_2(s) - f_1(s)} + O\left(\frac{f_2'(s) - f_1'(s)}{f_2(s) - f_1(s)}\right) \text{ as } s \to 0^+.$$

Example 1 (exponential cusp) : Let
$f_1(x) = pe^{-\frac{1}{x^2}}$, $f_2(x) = qe^{-\frac{1}{x^2}}$, with constants $p < q$. Then

$$u(x,y) = \frac{\cos\gamma_1 + \cos\gamma_2}{q - p} e^{\frac{1}{x^2}} + O(x^{-3}) \text{ as } x \to 0^+.$$

Under Condition A : $u(s,t) = \Theta\left(\frac{1}{f_2(s)-f_1(s)}\right)$ as $s \to 0^+$ i.e., there exist positive constans $k_1 < k_2$ so that
$\frac{k_1}{f_2(s)-f_1(s)} \leq |u(s,t)| \leq \frac{k_2}{f_2(s)-f_1(s)}$ for $0 < s < s_1$.

Example 2 (osculatory cusp, nonzero curvature) : Let
$f_1(x) = x^2 + px^3,\ f_2(x) = x^2 + qx^3,\ p < q$. Then
$u(x,y) = \Theta\left(\frac{1}{x^3}\right)$ as $x \to 0^+$.

Numerical investigation and a conjectue is given for the osculatory case in [Aoki and De Sterck 2014]. Further analysis is needed.

Case $\gamma_1 + \gamma_2 = \pi$

Suppose $f_1,\ f_2 \in C^2([0, x_0])$. A supersolution can be constructed to show that u is bounded near the origin. It follows from [Lancaster and Siegel 1996] that u is continuous at the origin.

Circular Cusp

[Yasunori Aoki 2007] Let Ω be bounded by two circles tangent to the x-axis at the origin: $\Omega = \{(p, q) : b < q < a,\ 0 < p < \infty\}$ where $p = \frac{x}{x^2+y^2}$, $q = \frac{y}{x^2+y^2}$, $a > 0$. $\Rightarrow x = \frac{p}{p^2+q^2}$, $y = \frac{q}{p^2+q^2}$

$Nu = u$ in Ω, $Tu \cdot \nu = \cos\gamma_1$ on $q = a$, $Tu \cdot \nu = \cos\gamma_2$ on $q = b$.

There is an approximate solution
$v(p, q) = Ap^2 - 2p\sqrt{1 - A^2(q - q_0)^2} - A(q - q_0)^2 + Aq_0^2$,
$A = \frac{\cos\gamma_1+\cos\gamma_2}{a-b}$, $q_0 = \frac{b\cos\gamma_1+a\cos\gamma_2}{\cos\gamma_1+\cos\gamma_2}$ with error estimate
$u(p, q) = v(p, q) + O(p^{-4})$ as $p \to \infty$.

v is the unique solution to $\nabla \cdot (\frac{\nabla v}{|\nabla v|}) = v$ in Ω , $\frac{\nabla v}{|\nabla v|} \cdot \nu = \cos\gamma$ on $\partial\Omega$.

Narrow Tube of General Cross-Section

Consider $Nu = Bu$ in Ω, $Tu \cdot \nu = \cos\gamma$ on $\partial\Omega$.
Let Ω be bounded. Suppose v satisfying

$$Nv = A, \text{ in } \Omega, \; Tv \cdot \nu = \cos\gamma \text{ on } \partial\Omega, \; A = \frac{\cos\gamma |\partial\Omega|}{|\Omega|},$$

$$\int_\Omega v(x)\,dx = 0$$

exists. Then $u = \frac{A}{B} + v + O(B)$ as $B \to 0$. [Siegel 1987], [Finn 1986]

[E. Miersemann 1993] gave a complete asymptotic series and [L.-F. Tam 1986] analyzed the case when v doesn't exist.

Now, let the complement of Ω be bounded.
Problem: Find the leading order behavior as B tends to zero.

[Siegel 1990] For large r, the solution behaves like a solution of the linearized problem $\Delta v = Bv$ in Ω:

$$u = \mathcal{K}_0(\sqrt{B}r)\left[c(\theta) + O\left(\frac{1}{r}\right)\right] \text{ as } r \to \infty.$$

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