C&O 330 - SOLUTIONS #2

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This assignment is about partitions. The questions are taken from the Exercises of Section 2.4 (page 38) of the Course Notes.

A **(15 points)** Question 2.

Solution: Let $\mathcal{P}$ be the set of all partitions, let $\lambda(\pi)$ be the number of 1s in π, let $\mu(\pi)$ be the number of distinct parts in $\pi \in \mathcal{P}$ and let $\omega(\pi) = n$ where $\pi \vdash n$. Let $A(x,u) = [(\mathcal{P}, \omega \otimes \lambda)]_o (x,u) = \sum_{n,k\geq 0} a_{n,k} x^n u^k$ and let $B(x,u) = [(\mathcal{P}, \omega \otimes \mu)]_o (x,u) = \sum_{n,k\geq 0} b_{n,k} x^n u^k$. Note that clearly $A, B \in \mathbb{C}[u][[x]]$. Then the total number of 1s in the set of all partitions of n is

$$c_n = \sum_{k=0}^{n} k a_{n,k} = [x^n] L_u \frac{\partial A}{\partial u}$$

and the sum of the number of distinct parts in the set of all partitions of n is

$$d_n = \sum_{k=0}^{n} k b_{n,k} = [x^n] L_u \frac{\partial B}{\partial u},$$

where $L_u f(x,u)$ is defined to be $f(x,1)$ for $f \in \mathbf{C}[u][[x]]$. Then

$$c_n = d_n \text{ for all } n \geq 0 \Longleftrightarrow L_u \frac{\partial A}{\partial u} = L_u \frac{\partial B}{\partial u}$$

so, to establish the result, we demonstrate that the right hand side of the bi-implicant holds.

A universal decomposition for $\mathcal{P}$ is

$$\Omega \colon \mathcal{P} \leftrightarrow \times_{i\geq 1} \{i\}^{\star}.$$

Clearly, Ω is additively $\omega \otimes \lambda$-preserving, so

$$A = \prod_{i\geq 1} \left[\left(\{i\}^{\star}\right), \omega \otimes \lambda\right]_o (x,u),$$

by the Product Lemma. But

$$\begin{aligned} \left[\left(\{i\}^{\star}\right), \omega \otimes \lambda\right]_o &= \sum_{j\geq 0} \left[\left(i^j, \omega \otimes \lambda\right)\right]_o = \sum_{j\geq 0} x^{\omega\left(i^j\right)} u^{\lambda\left(i^j\right)} \\ &= \begin{cases} 1 + ux + u^2x^2 + \cdots & \text{if} \quad i = 1, \\ 1 + x^i + x^{2i} + \cdots & \text{if} \quad i \neq 1, \end{cases} \\ &= \begin{cases} \frac{1}{1-ux} & \text{if} \quad i = 1, \\ \frac{1}{1-x^i} & \text{if} \quad i \neq 1, \end{cases} \end{aligned}$$

so

$$A = \frac{1}{1-ux}\prod_{i\geq 2}\frac{1}{1-x^i}.$$

Then

$$L_u\frac{\partial A}{\partial u} = \frac{x}{1-x}\prod_{i\geq 1}\frac{1}{1-x^i}.$$

Similarly,

$$B = \prod_{i\geq 1}\left[\left(\{i\}^\star\right), \omega\otimes\mu\right]_o(x,u)$$

where

$$\begin{aligned}\left[\left(\{i\}^\star\right), \omega\otimes\mu\right]_o(x,u) &= \sum_{j\geq 0} x^{\omega\left(i^j\right)}u^{\mu\left(i^j\right)} \\ &= 1+u\left(x^i+x^{2i}+\cdots\right) = \frac{1+x^i(u-1)}{1-x^i},\end{aligned}$$

so

$$B = \prod_{i\geq 1}\frac{1+x^i(u-1)}{1-x^i}.$$

Then

$$\log(B) = \sum_{i\geq 1}\left(1+x^i(u-1)\right) - \sum_{i\geq 1}\left(1-x^i\right),$$

and, differentiating partially with respect to u, we have

$$L_u\frac{1}{B}\frac{\partial B}{\partial u} = L_u\sum_{i\geq 1}\frac{x^i}{1+x^i(u-1)} = \sum_{i\geq 1}x^i = \frac{x}{1-x}.$$

Thus

$$L_u\frac{\partial B}{\partial u} = \frac{x}{1-x}L_u\frac{1}{B} = \frac{x}{1-x}\prod_{i\geq 2}\frac{1}{1-x^i} = L_u\frac{\partial A}{\partial u},$$

and the result follows.

B **(15 points)** Question 3.

Solution: The following proof is quite formal and it expresses precisely how the weight preservation is achieved. You now have the mathematical ideas to establish this clearly. Several decompositions are used that you have seen in the lectures. The same notation is used.

Let $\mathcal{A}$ be the set of all partitions in which no part appears exactly once. For $\pi\in\mathcal{A}$ and $\pi\vdash n$, let $\omega(\pi)=n$. By restricting the universal decomposition for the set of all partitions,

$$\mathcal{A}\leftrightarrow \times_{i\geq 1}\left(\{i\}^\star - \{i\}\right)$$

is an additively ω-preserving decomposition of $\mathcal{A}$. Then, by the Product Lemma,

$$\begin{aligned}
\left[(\mathcal{A},\omega)\right]_o(x) &= \prod_{i\geq 1}\left(\frac{1}{1-x^i}-x^i\right)=\prod_{i\geq 1}\frac{1-x^i+x^{2i}}{1-x^i}=\prod_{i\geq 1}\frac{1+x^{3i}}{1-x^{2i}}\\
&= \prod_{i\geq 1}\frac{1-x^{6i}}{\left(1-x^{2i}\right)\left(1-x^{3i}\right)}\\
&= \prod_{i\geq 1}\frac{1-x^{6i}}{\left(1-x^{6i}\right)\left(1-x^{6i-2}\right)\left(1-x^{6i-4}\right)\cdot\left(1-x^{6i}\right)\left(1-x^{6i-3}\right)}\\
&= \prod_{i\geq 1}\frac{1}{\left(1-x^{6i-2}\right)\left(1-x^{6i-4}\right)\left(1-x^{6i}\right)\left(1-x^{6i-3}\right)}\\
&= \prod_{i\geq 0}\frac{1}{\left(1-x^{6i+4}\right)\left(1-x^{6i+2}\right)\left(1-x^{6i+6}\right)\left(1-x^{6i+3}\right)}\\
&= \left[(\mathcal{B},\omega)\right]_o(x),
\end{aligned}$$

where B is the set of all partitions with no parts congruent to 1 or 5 mod 6.

C **(15 points)** Question 6.

Solution: Let $\mathcal{D}$ be the set of all partitions with distinct parts. Then each F_π, for $\pi\in\mathcal{D}$, has a maximal right angled isosceles triangle in the top left corner of F_π. Let k be the length of the two equal sides, and denote this triangle by T_k. Then, on deletion of T_k, a diagram remains that, when justified to the left, is a Ferrers diagram of a partition α with at most k parts. Then

$$\Omega\colon \mathcal{D}\leftrightarrow \dot{\bigcup_{k\geq 0}}\{T_k\}\times\mathcal{M}_k\colon F_\pi\mapsto (T_k,F_\alpha)\,.$$

Let $\omega(\pi)=n$, where $\pi\vdash n$, and $\theta(\pi)$ be the number of parts of π. Let o denote the zero weight function. Then

$$\begin{aligned}
(\omega\otimes\theta)(\pi) &= (\omega(\pi),\theta(\pi))=(n,k)=(\omega(T_k)+\omega(F_\alpha),\theta(T_k))\\
&= (\omega(T_k),\theta(T_k))+(\omega(F_\alpha),0)\\
&= (\omega(T_k),\theta(T_k))+(\omega(F_\alpha),o(F_\alpha))\\
&= ((\omega\otimes\theta)\oplus(\omega\otimes o))(T_k,F_\alpha)\\
&= ((\omega\otimes\theta)\oplus(\omega\otimes o))\,\Omega(\pi)
\end{aligned}$$

for all $\pi\in\mathcal{D}$, so

$$\omega\otimes\theta=((\omega\otimes\theta)\oplus(\omega\otimes o))\,\Omega.$$

If follows that Ω is an additively $\omega\otimes\theta$-preserving decomposition of $\mathcal{D}$ (the equality establishes $\omega\otimes\theta$-*preservation*, and the $\oplus$ indicates that this is done *additively*). But

$$\mathcal{D}\leftrightarrow \times_{i\geq 1}\{\varepsilon,i\}$$

so

$$[(\mathcal{D},\omega\otimes\theta)]_o(x,u)=\prod_{i\geq1}\left(x^{\omega(\varepsilon)}u^{\theta(\varepsilon)}+x^{\omega(i)}u^{\theta(i)}\right)=\prod_{i\geq1}\left(1+x^iu\right).$$

Similarly,

$$[(\{T_k\},\omega\otimes\theta)]_o(x,u)=u^kx^{k(k+1)/2}.$$

Also $\mathcal{M}_k\leftrightarrow\mathcal{L}_k\leftrightarrow\times_{i\geq1}\{i\}^\star$ is additively $\omega\otimes o$-preserving so

$$\begin{aligned}[(\mathcal{M}_k,\omega\otimes o)]_o(x,u) &= \prod_{i=1}^k\left[\left(\{i\}^\star,\omega\otimes o\right)\right]_o(x,u)\\ &= \prod_{i=1}^k\left(\sum_{j\geq0}x^{\omega\left(i^j\right)}u^{o\left(i^j\right)}\right)\\ &= \prod_{i=1}^k\left(\sum_{j\geq0}x^{ij}u^0\right)=\prod_{i=1}^k\frac{1}{1-x^i}.\end{aligned}$$

Combining this expressions, we have, by the Sum and Product Lemmas,

$$\begin{aligned}\prod_{i\geq1}\left(1+x^iu\right) &= [(\mathcal{D},\omega\otimes\theta)]_o(x,u)\\ &= \sum_{k\geq0}[(\{T_k\}\times\mathcal{M}_k,(\omega\otimes\theta)\oplus(\omega\otimes o))]_o\\ &= 1+\sum_{k\geq1}[(\{T_k\},\omega\otimes\theta)]_o[(\mathcal{M}_k,\omega\otimes o)]_o\\ &= 1+\sum_{k\geq1}\frac{u^kx^{k(k+1)/2}}{\prod_{i=1}^k(1-x^i)},\end{aligned}$$

completing the proof.

D **(15 points)** Question 8.

Solution: We consider first the right hand side of the result to be proved. It looks as if this involves the generating series for a maximal square of size k, a partition into at most k parts, and a partition with largest part at most k. We therefore surmise that this counts the set $\mathcal{P}$ of all partitions π with respect to the sum $\omega(\pi)$ of the parts of π, the number $\nu(\pi)$ of parts of π, and the size $\lambda(\pi)$ of the largest part of π. We now formalise this.

We use the universal decomposition

$$\Omega\colon\mathcal{P}\leftrightarrow\dot{\bigcup_{k\geq0}}\{D_k\}\times\mathcal{L}_k\times\mathcal{M}_k\colon\pi\mapsto(D_k,F_\alpha,F_\beta).$$

Now $\omega(\pi)=\omega(D_k)+\omega(F_\alpha)+\omega(F_\beta)$, $\nu(\pi)=\nu(D_k)+\nu(F_\alpha)+o(F_\beta)$, and $\lambda(\pi)=\lambda(D_k)+o(F_\alpha)+\lambda(F_\beta)$. Then

$$\begin{aligned}(\omega\otimes\nu\otimes\lambda)(\pi) &= (\omega\otimes\nu\otimes\lambda)(D_k)+(\omega\otimes\nu\otimes o)(F_\alpha)+(\omega\otimes o\otimes\lambda)(F_\beta)\\ &= ((\omega\otimes\nu\otimes\lambda)\oplus(\omega\otimes\nu\otimes o)\oplus(\omega\otimes o\otimes\lambda))(D_k,F_\alpha,F_\beta)\\ &= ((\omega\otimes\nu\otimes\lambda)\oplus(\omega\otimes\nu\otimes o)\oplus(\omega\otimes o\otimes\lambda))\,\Omega(\pi)\end{aligned}$$

for all $\pi \in \mathcal{P}$, so

$$\omega \otimes \nu \otimes \lambda = ((\omega \otimes \nu \otimes \lambda) \oplus (\omega \otimes \nu \otimes o) \oplus (\omega \otimes o \otimes \lambda))\, \Omega.$$

We conclude for this that Ω is an additively $\omega \otimes \nu \otimes \lambda$-preserving decomposition of $\mathcal{P}$. Then, by the Sum and Product Lemmas,

$$\begin{aligned}\left[(\mathcal{P}, \omega \otimes \nu \otimes \lambda)\right]_o (x,z,w) \quad &= \quad 1 + \sum_{k \geq 1} \left[(\{D_k, \omega \otimes \nu \otimes \lambda\})\right]_o \\ &\quad \cdot \left[(\mathcal{L}_k, \omega \otimes \nu \otimes o)\right]_o \cdot \left[(\mathcal{M}_k, \omega \otimes o \otimes \lambda)\right]_o .\end{aligned}$$

But

$$\left[(\{D_k, \omega \otimes \nu \otimes \lambda\})\right]_o (x,z,w) = x^{k^2} z^k w^k.$$

$$\begin{aligned}\left[(\mathcal{L}_k, \omega \otimes \nu \otimes o)\right]_o (x,z,w) &= \left[\left(\times_{i=1}^{k} \{i\}^{\star}, \omega \otimes \nu \otimes o\right)\right]_o \\ &= \prod_{i=1}^{k} \left(\sum_{j \geq 0} x^{\omega\left(i^j\right)} z^{\nu\left(i^j\right)} w^{o\left(i^j\right)} \right) \\ &= \prod_{i \geq 1} \left(\sum_{j \geq 0} x^{ij} z^j w^0 \right) = \prod_{i \geq 1} \frac{1}{1 - zx^i}.\end{aligned}$$

Similarly, using the decomposition

$$\tilde{}: \mathcal{M}_k \leftrightarrow \mathcal{L}_k,$$

we have, noting the effect of conjugation on the weight functions and the indeterminates,

$$\left[(\mathcal{M}_k, \omega \otimes o \otimes \lambda)\right]_o (x,z,w) = \left[(\mathcal{L}_k, \omega \otimes \nu \otimes o)\right]_o (x,w,z) = \prod_{i \geq 1} \frac{1}{1 - wx^i}.$$

Thus, combining these results, we have

$$\left[(\mathcal{P}, \omega \otimes \nu \otimes \lambda)\right]_o (x,z,w) = 1 + \sum_{k \geq 1} \frac{x^{k^2} z^k w^k}{\prod_{i=1}^{k} (1 - zx^i)(1 - wx^i)}.$$

We now determine $\left[(\mathcal{P}, \omega \otimes \nu \otimes \lambda)\right]_o (x,z,w)$ in another way, by deleting the largest part of π. Then π decomposed uniquely into a larges part k and a partition with largest part at most k. Thus

$$\Theta \colon \mathcal{P} \leftrightarrow \dot{\bigcup_{k \geq 0}} \{k\} \times \mathcal{L}_k \colon \pi \mapsto (k, F_\alpha).$$

Then $\omega(\pi) = \omega(k) + \omega(F_\alpha)$, $\nu(\pi) = \nu(k) + \nu(F_\alpha)$, and $\lambda(\pi) = \lambda(k) + o(F_\alpha)$ so

$$\begin{aligned}(\omega \otimes \nu \otimes \lambda)(\pi) &= (\omega \otimes \nu \otimes \lambda)(k) + (\omega \otimes \nu \otimes o)(F_\alpha) \\ &= ((\omega \otimes \nu \otimes \lambda) \oplus (\omega \otimes \nu \otimes o))(k, F_\alpha) \\ &= ((\omega \otimes \nu \otimes \lambda) \oplus (\omega \otimes \nu \otimes o))\, \Omega(\pi)\end{aligned}$$

for all $\pi \in P$, so

$$\omega \otimes \nu \otimes \lambda = (\omega \otimes \nu \otimes \lambda) \oplus (\omega \otimes \nu \otimes o)\, \Omega.$$

Thus, Θ is additively $\omega\otimes\nu\otimes\lambda$-preserving. Then, by the Sum and Product Lemmas,

$$\begin{aligned}[(\mathcal{P},\omega\otimes\nu\otimes\lambda)]_o(x,z,w) &= 1+\sum_{k\geq1}[(\{k\}\times\mathcal{L}_k,(\omega\otimes\nu\otimes\lambda)\oplus(\omega\otimes\nu\otimes o))]_o\\ &= 1+\sum_{k\geq1}[(\{k\},\omega\otimes\nu\otimes\lambda)]_o[\mathcal{L}_k,(\omega\otimes\nu\otimes o)]_o.\end{aligned}$$

But,

$$[(\{k\},\omega\otimes\nu\otimes\lambda)]_o = x^{\omega(k)}z^{\nu(k)}w^{\lambda(k)} = x^kzw^k.$$

Also

$$\begin{aligned}[\mathcal{L}_k,(\omega\otimes\nu\otimes o)]_o &= \left[\left(\times_{i=1}^k\{i\}^\star,\omega\otimes\nu\otimes o\right)\right]_o = \prod_{i=1}^k\left(\sum_{j\geq0}x^{\omega(i^j)}z^{\nu(i^j)}w^{o(i^j)}\right)\\ &= \prod_{i=1}^k\left(\sum_{j\geq0}x^{ij}z^jw^0\right) = \prod_{i=1}^k\frac{1}{1-zx^i}.\end{aligned}$$

Thus

$$[(\mathcal{P},\omega\otimes\nu\otimes\lambda)]_o(x,z,w) = 1+\sum_{k\geq1}\frac{x^kzw^k}{\prod_{i=1}^k(1-zx^i)}.$$

We conclude that

$$\sum_{k\geq1}\frac{x^{k^2}z^kw^k}{\prod_{i=1}^k(1-zx^i)(1-wx^i)} = \sum_{k\geq1}\frac{x^kzw^k}{\prod_{i=1}^k(1-zx^i)}.$$

E **(15 points)** Let q and t be indeterminates, and let $F_n(t,q)=\prod_{i=0}^n\left(1-tq^i\right)^{-1}$. This has a power series expansion of the form $F_n(t,q)=1+\sum_{k\geq1}t^kc_{k,n}(q)$, so

$$c_{k,n}(q) = \left[t^k\right]F_n(t,q).$$

The problem is to determine this coefficient. By considering an expression for $F_n(tq,q)$ that involves $F_n(t,q)$, prove that

$$c_{k,n}(q) = \prod_{i=1}^k\left(1-q^{n+i}\right)\left(1-q^i\right)^{-1}.$$

Solution: First, note that

$$F_n(tq,q) = \prod_{i=0}^n\frac{1}{1-tq^{i+1}} = \prod_{i=1}^{n+1}\frac{1}{1-tq^i} = F_n(t,q)\frac{1-t}{1-tq^{n+1}},$$

so $F_n(t,q)$ satsfies the functional equation

$$\left(1-tq^{n+1}\right)F_n(tq,q) = (1-t)F_n(t,q).$$

Then

$$\left(1-tq^{n+1}\right)\sum_{k\geq0}t^kq^kc_{k,n}(q) = (1-t)\sum_{k\geq0}t^kc_{k,n}(q),$$

where $c_{0,n}(q) = 1$, for $n \geq 0$. Applying $[t^m]$ to both sides, we have

$$q^m c_{m,n}(q) - q^{m+n} c_{m-1,n}(q) = c_{m,n}(q) - c_{m-1,n}(q)$$

whence, $(c_{m,n} \colon m = 0, 1, 2, \ldots)$ satisfies the linear recurrence relation

$$(1 - q^m)\, c_{m,n}(q) = \left(1 - q^{m+n}\right) c_{m-1,n}(q)$$

for $m > 0$, with initial condition $c_{0,n}(q) = 1$ for $n \geq 0$. Then

$$\frac{c_{m,n}(q)}{c_{m-1,n}(q)} = \frac{1 - q^{m+n}}{1 - q^m}$$

for $m > 0$, so

$$\frac{c_{M,n}(q)}{c_{0,n}(q)} = \prod_{m=1}^{M} \frac{1 - q^{m+n}}{1 - q^m}.$$

The result follows immediately.

F **Bonus**: **(15 points)** Find a natural bijection that accounts of the result given in Question A.