C&O 330 - SOLUTIONS #1

PROFESSOR D.M. JACKSON

(1) **(20 points)** Give an expression for $[x^n]\, G$ as an explicit function of n of each of the following two functional equations.

(a) **(10 points)** $G = F$, where F satisfies the functional equation

$$F = \frac{x}{(1-F)^m},$$

where m is a positive integer.

SOLUTION: By Lagrange's (Implicit Function) Theorem,

$$G = F = \sum_{n\geq 1} \frac{x^n}{n} \left[\lambda^{n-1}\right] (1-\lambda)^{-mn} = \sum_{n\geq 1} \frac{x^n}{n} \binom{(m+1)\, n-2}{n-1}.$$

(b) **(10 points)** $G = e^T$ where T satisfies the functional equation

$$T = xe^T.$$

SOLUTION: By Lagrange's (Implicit Function) Theorem,

$$\begin{aligned} G &= e^T = G = [x^n]\, e^T = 1 + \sum_{n\geq 1} \frac{x^n}{n} \left[\lambda^{n-1}\right] e^{\lambda} e^{n\lambda} \\ &= 1 + \sum_{n\geq 1} \frac{x^n}{n} \left[\lambda^{n-1}\right] e^{(n+1)\lambda} \\ &= 1 + \sum_{n\geq 1} \frac{x^n}{n!} (n+1)^{n-1}. \end{aligned}$$

Comment: Other solutions : i) By Lagrange's Theorem,

$$\begin{aligned} T(x) &= \sum_{n\geq 1} \frac{x^n}{n} \left[\lambda^{n-1}\right] e^{n\lambda} = \sum_{n\geq 1} n^{n-1} \frac{x^n}{n!} \\ &= x \sum_{n\geq 1} n^{n-1} \frac{x^{n-1}}{n!}, \end{aligned}$$

so

$$xe^T = x \sum_{n\geq 1} n^{n-1} \frac{x^{n-1}}{n!}.$$

Then, by the Cancellation Law, or by observing that

$$x \left(e^T - \sum_{n\geq 1} n^{n-1} \frac{x^{n-1}}{n!} \right) = 0$$

and that $\mathbb{C}[[x]]$ has no zero divisors, we conclude that

$$e^T - \sum_{n\geq 1} n^{n-1}\frac{x^{n-1}}{n!} = 0$$

so

$$e^T = \sum_{n\geq 1} n^{n-1}\frac{x^{n-1}}{n!} = \sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!}.$$

Zero divisors are mentioned in Math 235, in connexion with the ring of square matrices.
ii) Alternatively, x^{-1} can be used as long as it is clear that the ring being used is the ring of Laurent series over $\mathbb{C}$. This ring was mentioned in the lectures. Half credit should be given for a 'solution' in $\mathbb{C}[[x]]$ that uses x^{-1}, for I have emphasised the importance of working accurately with formal power series.

(2) **(20 points)** A regular n-gon (that is, a regular polygon with n sides) A in the plane has a distinguished edge. A *diagonal* of A is a line segment, in the interior of A, joining two vertices of A. Find the number, c_n, of dissections of A into triangles by diagonals that meet only at vertices of A.

Solution: Let $\mathbb{D}$ be a dissection of A. Let $\mathbb{D}^\circ$ be the dual of $\mathbb{D}$. Now split the vertex of $\mathbb{D}^\circ$ associated with the external face of $\mathbb{D}$ into n vertices and, of these, select the monovalent vertex associated with the root edge of $\mathbb{D}$ as the root vertex of the tree $\mathbb{T}$ thus formed. $\mathbb{T}$ is easily characterised as a plane planted cubic tree with n non-root monovalent vertices. This construction is bijective since the dual of $\mathbb{D}^\circ$ is $\mathbb{D}$. Let $\mathcal{T}$ be the set of all plane planted trees and let $\omega(t)$ be the number of non-root monovalent vertices of $t \in \mathcal{T}$. Let $T(x) = [(\mathcal{T},\omega)]_o$. Thus, by construction,

$$c_n = \left[x^{n-1}\right] T$$

where $T(x)$ satisfies the functional equation

$$T = x + T^2.$$

Then

$$T = \frac{x}{1-T},$$

which is a functional equation in Lagrangian form. Then, by Lagrange's Theorem,

$$c_n = \frac{1}{n-1}\left[x^{n-2}\right](1-\lambda)^{-(n-1)} = \frac{1}{n-1}\binom{2(n-2)}{n-2}.$$

(3) **(20 points)** Find the number, a_n, of plane planted trees on n non-root vertices with no vertices of degree 2. Your answer should give a_n as an explicit function of n.

Solution: Let $\mathcal{P}$ be the set of all plane planted trees, and let $\mathcal{S}$ be the set of all trees in $\mathcal{P}$ with no vertices of degree 2. By the Branch Decomposition for plane planted trees, we have

$$\Omega\colon \mathcal{P} \longleftrightarrow \{\uparrow\} \times \cup_{k\geq 0}\mathcal{P}^k,$$

where the union is disjoint (and $\longleftrightarrow$ indicates bijectivity). Then, clearly,

$$\Omega|_{\mathcal{S}} : \mathcal{S} \longleftrightarrow \{\uparrow\} \times \cup_{k\geq 0, k\neq 1}\mathcal{S}^k.$$

Let $\omega(t)$ be the number of non-root vertices in $t \in \mathcal{S}$, and let $S(x) = [(\mathcal{S}, \omega)]_o$. Now $\Omega|_{\mathcal{S}}$ is additively ω-preserving on $\mathcal{S}$ so, by the Sum and Product Lemmas,

$$S = x\left(1 + S + S^2 + S^3 +\right) = x\left(1 + \frac{S^2}{1-S}\right).$$

Thus, by Lagrange's Theorem, S is the unique solution of the functional equation

$$S = x\left(1 + \frac{s^2}{1-S}\right).$$

in $\mathbb{C}[[x]]$. Again, by Lagrange's Theorem,

$$\begin{aligned}
a_n &= [x^n] S(x) = \frac{1}{n}\left[\lambda^{n-1}\right]\left(1 + \frac{\lambda^2}{1-\lambda}\right)^n \\
&= \frac{1}{n}\left[\lambda^{n-1}\right]\sum_{i\geq 0}\binom{n}{i}\lambda^{2i}(1-\lambda)^{-i} \\
&= \frac{1}{n}\left[\lambda^{n-1}\right]\sum_{i\geq 0}\binom{n}{i}\lambda^{2i}\sum_{j\geq 0}\binom{i+j-1}{j}\lambda^j \\
&= \frac{1}{n}\sum_{i,j\geq 0, j=n-1-2i}\binom{n}{i}\binom{i+j-1}{j} \\
&= \frac{1}{n}\sum_{i,j\geq 0}\binom{n}{i}\binom{n-i-2}{i-1}.
\end{aligned}$$

Then, in the summation, $i \geq 1$ and $n-i-2 \geq i-1$ so $i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Thus

$$c_n = \frac{1}{n}\sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\binom{n}{i}\binom{n-i-2}{i-1}.$$

(4) **(20 points)**

(a) **(10 points)** Solve the functional equation

$$P = x\left(\frac{1}{1-P} + (u-1)P\right),$$

for $P \equiv P(x,u)$, where x and u are indeterminates, by giving $\left[x^n u^k\right] P(x,u)$ as an explicit expression in n and k.

Solution: By Lagrange's Theorem,

$$\begin{aligned}
\left[x^n u^k\right] P(x,u) &= \left[u^k\right] \frac{1}{n} \left[\lambda^{n-1}\right] \left(\frac{1}{1-\lambda} + (u-1)\lambda\right)^n \\
&= \frac{1}{n} \left[\lambda^{n-1}\right] \left[u^k\right] \sum_{i=0}^{n} \binom{n}{i} (u-1)^i \lambda^i (1-\lambda)^{-(n-j)} \\
&= \frac{1}{n} \left[\lambda^{n-1}\right] \sum_{i=0}^{n} \binom{n}{i} \left(\left[u^k\right](u-1)^i\right) \lambda^i (1-\lambda)^{-(n-i)} \\
&= \frac{1}{n} \left[\lambda^{n-1}\right] \sum_{i=k}^{n} \binom{n}{i} \binom{i}{k} (-1)^{i-k} \lambda^i (1-\lambda)^{-(n-i)} \\
&= \frac{1}{n} \sum_{i=k}^{n} \binom{n}{i} \binom{i}{k} (-1)^{i-k} \left[\lambda^{n-1-i}\right] (1-\lambda)^{-(n-i)} \\
&= \frac{1}{n} \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{i} \binom{i}{k} \binom{2(n-i-1)}{n-i-1}.
\end{aligned}$$

Comments: a) The operators $[x^n]$ and $\left[u^k\right]$ commute as do $\left[u^k\right]$ and $\left[\lambda^{n-1}\right]$, since they are representable in terms of differential with respect to different indeterminates. b) The Binomial Series Theorem has been used. This is familiar from Math 239. c) Negative binomials have been changed to standard binomials. This is also familiar from Math 239.

(b) **(10 points)** Construct a set $\mathcal{P}$ of *combinatorial structures* and a weight function ω for $\mathcal{P}$ such that

$$\left[(\mathcal{P},\omega)\right]_o = P(x,u).$$

Note that the weight function ω must be of the form

$$\omega\colon \mathcal{P} \to \{0,1,2,\ldots\}^2 : \sigma \mapsto (\omega_1(\sigma), \omega_2(\sigma))$$

where

$$\omega_1, \omega_2\colon \mathcal{P} \to \{0,1,2,\ldots\}.$$

and the form of the generating series P is

$$P(x,u) = \sum_{\sigma\in\mathcal{P}} x^{\omega_1(\sigma)} u^{\omega_2(\sigma)}.$$

(**Remark:** You have been introduced to bivariate generating series (generating series in two indeterminates) in Math 239.)

Solution: Now P satisfies the functional equation

$$P = x\left(\frac{1}{1-P} + (u-1)P\right),$$

which may be rewritten as

$$\begin{aligned}
P &= x\left(\left(1+P+P^2+P^3+\cdots\right) + (u-1)P\right) \\
&= x\left(1+uP+P^2+P^3+\cdots\right),
\end{aligned}$$

so P satisfies the functional equation

$$P = x\left(1+uP+P^2+P^3+\cdots\right).$$

Thus $\mathcal{P}$ decomposes as

$$\Omega\colon \mathcal{P} \longleftrightarrow \{\uparrow\} \times \cup_{k\geq 0}\mathcal{P}^k.$$

Since this is the Branch Decomposition for plane planted trees, we identify $\mathcal{P}$ as the set of all plane planted trees. Moreover, from the revised functional equation, the weight function ω on $\mathcal{P}$ evidently has the property that

$$\left[\left(\mathcal{P}^k, \omega\right)\right]_o = \begin{cases} xP^k & \text{if} \quad k \neq 1, \\ xuP^k & \text{if} \quad k = 1. \end{cases}$$

Then x marks non-root vertices, and u marks vertices with up-degree 1 or, equivalently, degree 2. In other words, for $\sigma \in \mathcal{P}$, $\omega_1(\sigma)$ is identified as the number of non-root vertices of σ, and $\omega_2(\sigma)$ is the number of vertices of degree 2 in σ. Thus, we deduce that $\left[x^n u^k\right] P(x, u)$ is the number of plane planted trees with n non-root vertices and k vertices of degree 2.