C&O 330 NOTES ON THE THE HARDY-RAMANUJAN FORMULA FOR $p(n)$

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1. An exact formula for the partition number $p(n)$

The following material is **not part of the course**, but is made available on the website because a few of you have asked me about it.

It is a note on an exact formula for the partition number $p(n)$. The derivation uses results from analysis that you will not have seen yet, so I am not including the proof. Those of you who are interested in tracking it down should try to read Chapter 14 of "Topics in Analytic Number Theory" by Hans Rademacher, published by Springer in 1973. It is in the Library!

The formula is due to Hardy and Ramanujan, and is referred to as the *Hardy-Ramanujan formula*. For $x \in \mathrm{R}$, let $[x]$ denote the integer part of x and , for integers h and k, let (h,k) denote the greatest common divisor of h and k. Then

$$p(n) = [t(n)]$$

where

(1)

$$t(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{[2\sqrt{n}/3]} \sqrt{k} A_k(n) \frac{d}{dn} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}$$

$$A_k(n) = \sum_{0<h\leq k,(h,k)=1} e^{\pi i s(h,k)} e^{-2\pi i h n/k},$$

(2) $s(h,k)$ is the Dedekind sum

$$s(h,k) = \sum_{j=1}^{k-1} \frac{j}{k} \left(\left(\frac{hj}{k}\right)\right),$$

(3)

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

Notice that $p(n)$ is given as the integer part of a *finite* expression that involves the transcendental number π as a factor.

It would be interesting if someone programmed this formula to test it on a few values of n.

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