C&O 330 - ASSIGNMENT #5

DUE FRIDAY, 19 DECEMBER AT 10:31PM

(1) **(20 points)**
  (a) **(15 points)** Let $\mathsf{u} \leftrightarrow <$ and $\mathsf{d} \leftrightarrow \geq$. Let $a_1, \ldots, a_4$ be positive integers. Find the number of permutations with pattern

$$\mathsf{u}^{a_1-1}\mathsf{d}\mathsf{u}^{a_2-1}\mathsf{d}\mathsf{u}^{a_3-1}\mathsf{d}\mathsf{u}^{a_4-1},$$

  expressing your result as a determinant of binomial numbers.
  Solution: This part is a slight generalisation of the similar question on the previous assignment. It is done in the same way, and many of the details are in the Notes! The result is

$$\det\left(\mathbf{M}_4\right)$$

  where

$$\mathbf{M}_4 = \left[\binom{\sum_{k=1}^{5-i} a_k}{\sum_{k=5-j}^{5-i} a_k}\right]_{4\times 4}$$

  (b) **(5 points)** On the basis of this evidence state a conjecture of the number of permutations with pattern

$$\mathsf{u}^{a_1-1}\mathsf{d}\mathsf{u}^{a_2-1}\mathsf{d}\cdots\mathsf{u}^{a_m-1}\mathsf{d}\mathsf{u}^{a_{m+1}-1}.$$

  Solution: The conjecture is

$$\det\left(\mathbf{M}_{m+1}\right)$$

  where

$$\mathbf{M}_{m+1} = \left[\binom{\sum_{k=1}^{m+2-i} a_k}{\sum_{k=5-j}^{m+2-i} a_k}\right]_{(m+1)\times(m+1)}.$$

(2) **(20 points)** A *dodecahedron* is a regular solid consisting of 12 regular pentagons arranged so that each vertex of the dodecahedron is incident with 3 pentagons and that every edge of the dodecahedron is incident with 2 pentagons.
  (a) **(15 points)** Find the cycle index polynomial for the automorphism group of the dodecahedron acting on the 12 faces of the dodecahedron.
  Solution: The automorphisms come from rotations around axes specified by **a)** two antipodal vertices, **b)** two andipodal edges and **c)** two faces.
  Case (a): There are 20 vertices, and therefore 10 pairs of antipodal vertices and therefore 10 axes of rotation passing through antipodal vertices..If c is a rotation through 120 degrees, then c and c^2 are automorphisms. Thus there are 20 automorphisms, each consisting of 4

3-cycles. The contribution of these to the cycle index polynomial is $20x_3^4$.

Case (b): The sum of the vertex degrees is 60, so there are 30 edges, and therefore 15 pairs of antipodal edges and therefore 15 axes of rotation passing through the midpoints of antipodal edges. If c is a rotation through 180 degrees about one of these axes, then c is an automorphism giving 1 automorphisms for each axis. Thus there are 15 automorphisms, each consisting of 6 2-cycles. The contribution of these to the cycle index polynomial is $15x_2^6$.

Case (c): There are 12 faces, and therefore 6 pairs of antipodal faces and therefore 6 axes of rotation passing through the midpoints of antipodal faces. If c is a rotation through 72 degrees about one of these axes, then c^i is an automorphism for $i = 1, \ldots, 4$, giving 4 automorphisms for each axis. Thus there are 24 automorphisms, each consisting of 2 1-cycles and 2 5-cycles.. The contribution of these to the cycle index polynomial is $24x_1^2x_5^2$.

Case (d): The identity. This gives x_1^{12}.

Combining these cases we find that there are 60 automorphisms. We can check this independently by the Orbit-Stabiliser Theorem. Let v be a vertex of the regular dodecahedron D, and let $G = \mathrm{aut}\,(D)$. Then $|G_x| = 3$ and $|Gx| = 20$, the number of vertices of D. Then $|G| = |G_x| \cdot |Gx| = 60$. The cycle index polynomial of G acting on the faces of D is therefore

$$Z_{\mathrm{aut}(D,f)}\,(x_1, \ldots, x_5) = \frac{1}{60}\left(x_1^{12} + 20x_3^4 + 15x_2^6 + 24x_1^2x_5^2\right).$$

(b) **(5 points)** Find the generating series for the number of ways of painting the faces of the dodecahedron with two colours.

Solution: Let r and g mark red faces and green faces, respectively. Then, by Pólya's Theorem, the generating series for the number of paintings of D with colours r and g is

$$F\,(r, g) = Z_{\mathrm{aut}(D,f)}\,\langle r + g\rangle\,.$$

Thus the required number is

$$\begin{aligned} F\,(1,1) &= Z_{\mathrm{aut}(D,f)}\,(2, \ldots, 2) \\ &= \left(2^{12} + 20\left(2^4\right) + 15\left(2^6\right) + 24\left(2^4\right)\right)/60 \\ &= 96. \end{aligned}$$

Also, $F\,(r, g) = Z_{\mathrm{aut}(D,f)}\left(r + g, r^2 + g^2, \ldots, r^5 + g^5\right)$ so

$$\begin{aligned} F\,(r, g) = \;& 5r^9g^3 + 24r^6g^6 + 5r^3g^9 + 3r^{10}g^2 + 12r^8g^4 + \\ & 12r^4g^8 + 3r^2g^{10} + 14r^7g^5 + r^{11}g + rg^{11} + \\ & 14g^7r^5 + r^{12} + g^{12}. \end{aligned}$$

so there are 3 ways of painting D with 2 red faces and 10 green faces.

(c) **(5 points)** Confirm the coefficient giving the number of ways of painting the faces of the dodecahedron with 2 red faces and 10 green faces by generating the paintings combinatorially.

Solution: The dodecahedron consists of two antipodal faces and two rings of 5 pentagonal faces with this axis as axis of symmetry. Select a face and therefore an axis of symmetry. Paint this face red. The only choices for the other face are 1) a face in the first pentagonal ring (one choice), 2) a face in the second pentagonal ring (1 choice) and 3) the antipodal face (1 choice). These are clearly distinct since they differ by the length of a shortest path between the two selected faces.This accounts for three distinct choices.

(3) **(20 points)** The following question is quite a long exercise in the pattern algebra and it will take time.

(a) **(10 points)** Find the generating series F for the set of all permutations with pattern $\left\{<^2\geq^2\right\}^\star$.

Solution: Let $h = 1 - u^2 d^2$. Then $F = \Delta\chi\phi h^{-1}$. Now, by a rightmost expansion of h, we have

$$\begin{aligned} h &= 1 - u^4 - u^2 dw + u^2 wu \\ &= q + LwR^t \end{aligned}$$

where $q = 1 - u^4$, $L = \left[-u^2 d, u^2\right]$, $R^t = [1, u]$. Then, by the Generalised Maximal Decomposition Theorem,

$$\phi\left(Rh^{-1}\right) = \left(I + \phi\left(Rq^{-1}L^t\right)\right)^{-1}\phi\left(q^{-1}L\right).$$

Let $\Xi = \Delta\chi\phi$. Then

$$\Xi\left(Rh^{-1}\right) = \left(I + \Xi\left(Rq^{-1}L^t\right)\right)^{-1}\Xi\left(q^{-1}L\right).$$

Then, by Crámer's Rule,

$$F = \frac{\begin{vmatrix} F_0 & F_2 \\ F_1 & 1 + F_3 \end{vmatrix}}{\begin{vmatrix} 1 + F_3 & F_2 \\ F_o & 1 + F_3 \end{vmatrix}},$$

where $F_k = \Xi\left(q^{-1}u^k\right)$. Each of these elements may be determined by the Permutation Lemma, and the details are straightforward and repetitive. The result is

$$F = \frac{\tan(x) + \tanh(x)}{1 + \sec(x)\sec\mathrm{h}(x)}.$$

(b) **(10 points)** Use a series of differential constructions to obtain a system of simultaneous differential equations for F.

[**Hint:** Delete the largest element of such a permutation, which then decomposes into two constituents, one of which may be null. Characterise the patterns of the two constituents. Repeat the differential process on the two new sets, which then introduces further sets. Ultimately you will observe than no new sets are introduced and that you therefore have a closed system of equations. That is to say, the process is naturally terminating.]

Solution: Use the Derivative Lemma to delete the biggest element in a permutation. A permutation will decompose into an ordered pair of permutations with the same repeating but perhaps different initial and terminal elements in the pattern. Since the repeating element is finite, there will be a finite number of different patterns modulo the repeating element. This will give a set of simultaneous differential equations. The patterns so obtained are $\left(<^2\geq^2\right)^\star$, $\left(<^2\geq^2\right)^\star <$, $\geq \left(<^2\geq^2\right)^\star$, $\geq \left(<^2\geq^2\right)^\star <$. Let f, a, b, c be the corresponding exponential generating series. Then

$$\begin{aligned}
\frac{d}{dx}f &= ab + 1,\\
\frac{d}{dx}a &= ac + f,\\
\frac{d}{dx}b &= c^2 + a + b,\\
\frac{d}{dx}c &= bc + f.
\end{aligned}$$

[It is in fact possible to solve this system by linearising it by a particular transformation.]

[**Comment:** The generating series F determined in part (a) from the pattern algebra is a solution of this system of equations. Such systems of equations are called *matrix Ricatti* equations.]