# C&O 330 EXTRA PROBLEMS
# COMBINATORIAL ENUMERATION

INSTRUCTOR: D.M.JACKSON

**I have collected together some questions for review purposes. Many you will have seen before, but you should try to do them without looking at any other material - at least in the first place. A few of them will be new to you.**

Here are some tips for reviewing the material of the course. In the lectures I have covered partitions of integers and sets, the Pattern Algebra, the Maximal Decomposition Theorem, Pólya's Theorem, the theory of exponential generating series and simple applications of Pólya's Theorem, and a substantial number of examples of the use of this material. These were the main topics.

(1) Proofs
  - (a) You should understand the proofs of Lagrange's Implicit Function Theorem and Pólya's, but I shall not ask you to prove them.
  - (b) You should be able to prove results such as the Permutation Lemma and the Maximal Decomposition Theorem.

(2) Using the theory
  - (a) You should be able to use the theory of partitions, the theory of exponential generating series, the Pattern Algebra, the Maximal Decomposition Theorem and Lagrange's Implicit Function Theorem and Pólya's Theorem to solve enumerative questions of the sort seen in the course.
  - (b) I shall not be asking you questions that involve the sophisticated double or reverse uses of Lagrange's Theorem that we saw in a few identities.
  - (c) You should be able to provide a good level of explanation in your solutions. Such explanations are a vital part of your solutions, and you will be given credit for them.
  - (d) You should understand the main combinatorial decompositions for permutations, partitions of sets, trees, functions, matrices, partitions of integers, Ferrers diagrams and so on.

(3) You should reread the solutions for the assignments that I have placed on the website. You should be able to understand all of the details that are given there.

(4) You should reread the relevant sections of the Course Notes that contain the material discussed in class. You will already be aware of the fact that these Notes contain more than I have covered in class. The additional material is there for people who wish to see some more complex applications of the material.

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## Problems - NOT to be handed in

(1) Find the generating series for the number of paths on the integer sublattice of the real plane that start at $(0,0)$ and end at $(n,n)$, with steps that are unit line segments in the positive $Ox$ or $Oy$ direction, and that do not meet the line $x=y$ at points with odd coordinates.

(2) The partitions of 5 are $5, 4\,1, 3\,2, 3\,1\,1, 2\,2\,1, 2\,1\,1\,1, 1\,1\,1\,1\,1$. The total number of "1"s in this set is 12. Now look at each partition one by one and count the number of distinct symbols in each partition. For the seven partitions of 5 listed above these numbers are $1,2,2,2,2,2,1$. The sum of these is again 12. Prove that this is true in general.

(3) Let $q$ and $t$ be indeterminates, and let $F_n(t,q) = \prod_{i=0}^{n}\left(1-tq^i\right)^{-1}$. This has a power series expansion of the form $F_n(t,q) = 1+\sum_{k\geq 1} t^k c_{k,n}(q)$. By considering an expression for $F_n(tq,q)$ that involves $F_n(t,q)$, prove that

$$c_{k,n}(q) = \prod_{i=1}^{k}\left(1-q^{n+i}\right)\left(1-q^i\right)^{-1}.$$

(4) **(10 marks total)**

(a) **(5 marks)** Let $\mathcal{D}$ be the set of all derangements, $\mathcal{P}$ the set of all permutations on $\{1,\ldots,n\}$ for $n=0,1,2,\ldots$ and let $\mathcal{C}$ be the set of all cycles of length $n$ on $\{1,\ldots,n\}$ for $n=1,2,\ldots$. Prove that

$$\frac{d\mathcal{D}}{d\mathsf{s}} \stackrel{\sim}{\rightarrow} \left(\frac{d}{d\mathsf{s}}\left(\mathcal{C}-\{1\}\right)\right)\star\mathcal{D},$$

where $\mathsf{s}$ denotes the generic $\mathsf{s}$-object.

(b) **(5 marks)** Use the disjoint cycle decomposition to show that the exponential generating series for $\mathcal{C}$ is $\log\left(1-x\right)^{-1}$. Hence obtain an explicit differential equation for the exponential generating series $D(x)$ for $\mathcal{D}$.

(5) **(5 marks)** Let $\mathcal{A}$ and $\mathcal{B}$ be sets of labelled configurations. Show combinatorially that

$$\frac{d}{d\mathsf{s}}(\mathcal{A}\star\mathcal{B}) \widetilde{\rightarrow} \left(\frac{d\mathcal{A}}{d\mathsf{s}}\star\mathcal{B}\right)\cup\left(\mathcal{A}\star\frac{d\mathcal{B}}{d\mathsf{s}}\right).$$

(6) **(10 marks)**

(a) **(5 marks)** Show that the number of permutations on $\{1,\dots,n\}$ with $k$ cycles is given by

$$\left[u^k\frac{x^n}{n!}\right](1-x)^{-u}.$$

(b) **(5 marks)** An *upper record* in a permutation $\pi$ on $\mathcal{N}_n$ is an element $i\in\mathcal{N}_n$ such that $\pi_i>\pi_j$ for all $j<i$. Show that the number of permutations on $\mathcal{N}_n$ with $k$ upper records is

$$\left[u^k\frac{x^n}{n!}\right](1-x)^{-u}.$$

(a) **(10 marks)** Let $a(K,M,N)$ be the number of $M\times N$ $\{0,1\}$-matrices with no rows or columns of zeros and with a total of $K$ ones. Let

$$A(u,x,y)=\sum_{k,m,n\geq 0}a(k,m,n)\,u^k\frac{x^m}{m!}\frac{y^n}{n!}.$$

It is known that $A$ satisfies the formal partial differential equation

$$(1+u)\frac{\partial A}{\partial u}=xy\left(1+\frac{\partial}{\partial x}\right)\left(1+\frac{\partial}{\partial y}\right)A.$$

Give a purely combinatorial proof of this equation using combinatorial properties of the derivative.

(7) **(10 marks)**

(a) **(5 marks)** Find the number of plane planted trees on $n$ non-root vertices, with no vertex having positive even up-degree.

(b) **(5 marks)** Find the number of plane planted trees on $n$ non-root vertices, with no vertex having positive even up-degree, such that the vertex adjacent to the root has up-degree $a$, where $a$ is an odd integer.

(8) **(10 marks)** Find the number of rooted labelled trees on $n$ vertices with no vertices of (total) degree 2.

(9) **(10 marks)** Let $f$ be a function $f\colon\mathcal{N}_n\to\mathcal{N}_n$, where $\mathcal{N}_n=\{1,\dots,n\}$. Let $f^{[m]}$ denote the $m$-fold composition of $f$ with itself, so $f^{[2]}=f\circ f$. For any $i\in\mathcal{N}_n$, the sequence $\left(f^{[m]}(i) : m=0,1,2,\dots\right)$ is eventually

periodic on a subset of $\mathcal{N}_n$. The elements of this subset are called *recurrent* elements of $f$.

For example, if $g\colon \mathcal{N}_{10} \to \mathcal{N}_{10}$ is the function given by

$$\begin{array}{lllll} g(1)=5, & g(2)=8, & g(3)=10, & g(4)=6, & g(5)=10, \\ g(6)=4, & g(7)=10, & g(8)=6, & g(9)=1, & g(10)=1, \end{array}$$

then $g(9)=1, g(1)=5, g(5)=10, g(10)=1$, then

$$\left(g^{[m]}(9) : m=0,1,2,\ldots\right) = (9,1,5,10,1,5,10,1,\ldots),$$

so $1,5,10$ are recurrent elements. Also, $g(2)=8, g(8)=6, g(6)=4, g(4)=6$, then

$$\left(g^{[m]}(2) : m=0,1,2,\ldots\right) = (2,8,6,4,6,4,6,4,\ldots),$$

so $4,6$ are also recurrent elements. Similarly, $g(3)=10, g(10)=1, g(1)=5, g(5)=10$ so we get no new recurrent elements; and $g(7)=10$, so we get no new recurrent elements. The complete set of recurrent elements of $g$ is therefore $\{1,4,5,6,10\}$.

Let $\mathcal{F}$ be the set of all functions $f\colon \mathcal{N}_n \to \mathcal{N}_n$ for $n=0,1,2,\ldots$.By considering the (familiar) decomposition

$$\mathcal{F} \overset{\sim}{\to} \mathcal{U} \circledast \mathcal{C} \circledast \mathcal{T}$$

where $\mathcal{C}$ is the set of all non-null cycles and $\mathcal{T}$ is the set of all rooted labelled trees, show that the number of functions $f\colon \mathcal{N}_n \to \mathcal{N}_n$ with $k$ recurrent elements is

$$k!\binom{n-1}{k-1}n^{n-k}.$$

(10) **(15 marks)**

(a) **(5 marks)** Let $a$ be an indeterminate. Find an expression for

$$\sum_{i\geq 1}\frac{1}{i}\binom{ai}{i-1}t^i$$

in terms of $w$, where $w \equiv w(t)$ is the unique solution of the functional equation

$$w = t(1+w)^a.$$

[**Hint:** Use the first form of Lagrange's Implicit Function Theorem.]

(b) **(5 marks)** Find an expression for

$$\sum_{j\geq 0}\binom{aj}{j}t^j$$

in terms of $w$, where $w$ is defined in part (a).

[**Hint:** Use the second form of Lagrange's Implicit Function Theorem.]

(c) **(5 marks)** Use parts (a) and (b) to prove that

$$\sum_{i=1}^{n} \frac{1}{i} \binom{ai}{i-1} \binom{a(n-i)}{n-i} = \binom{an}{n-1}.$$

[**Hint:** Use the second form of Lagrange's Implicit Function Theorem.]

(11) **(10 marks)** Find the formal power series $f$ that satisfies

$$(1-t)^3 f\left(t(1-t)^3\right) = 1.$$

(12) **(10 marks)** Show that the number of sequences on $\{1,\ldots,n\}$ of length $m$ and containing $k$ different symbols is

$$\binom{n}{k} \left[\frac{x^m}{m!}\right] (e^x - 1)^k.$$

(13) **(15 marks)**

(a) **(5 marks)** Show that the number of vertex coverings of the complete graph $K_n$ on $n$ vertices with cycles of length 3 or more is

$$\left[\frac{x^n}{n!}\right] (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{4}\right).$$

(b) **(5 marks)** Consider $n$ lines in general position in the plane, so no three of them are concurrent. A *frame* consists of $n$ of the $\binom{n}{2}$ points of intersection such that no three of the $n$ points lie on the same line. Show that the number $f_n$ of frames on $n$ lines is given by

$$f_n = \left[\frac{x^n}{n!}\right] (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{4}\right).$$

(c) **(5 marks)** Find a linear recurrence equation for $(f_n \colon n = 0,1,2,\ldots)$.

(14) **(Total: 30 marks)** The pattern $\nearrow\searrow\nearrow\searrow\nearrow\searrow$ is encoded as $(\mathsf{ud})^3$ where $\mathsf{u} \leftrightarrow <$ and $\mathsf{d} \leftrightarrow \geq$. Let $\operatorname{seq}(\mathsf{p})$ be the set of all sequences with pattern $\mathsf{p}$. The purpose of this question is to determine the generating series $\left[\operatorname{seq}\left((\mathsf{ud})^3\right)\right]_o$ for $\operatorname{seq}\left((\mathsf{ud})^3\right)$, and thence to determine the number $c_7$ of permutations on $\{1,\ldots,7\}$ with this pattern.

(a) **(5 marks)** Let $\mathsf{u}+\mathsf{d}=\mathsf{w}$. Prove in a systematic way, rather than by explicit expansion, that

$$(\mathsf{ud})^3 = \mathsf{uw}\,(\mathsf{ud})^2 - \mathsf{u}^3\mathsf{w}\,(\mathsf{ud}) + \mathsf{u}^5\mathsf{w} - \mathsf{u}^6.$$

(b) **(5 marks)** Let $f_k = \left[\mathrm{seq}\left((\mathsf{ud})^k\right)\right]_o$. Use part (a) to prove that the generating series $f_0,\ldots,f_3$ satisfy the system of linear equations

$$\begin{bmatrix} 1 & -\gamma_2 & \gamma_4 & -\gamma_6 \\ 0 & 1 & -\gamma_2 & \gamma_4 \\ 0 & 0 & 1 & -\gamma_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_3 \\ f_2 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} -\gamma_7 \\ \gamma_5 \\ -\gamma_3 \\ \gamma_1 \end{bmatrix}.$$

where $\gamma_k = \left[\mathrm{seq}\left(\mathsf{u}^{k-1}\right)\right]_o$, the generating series for all sequences with pattern $\mathsf{u}^{k-1}$.

(c) **(5 marks)** Use part (b) to prove that

$$f_3 = \begin{vmatrix} \gamma_2 & \gamma_4 & \gamma_6 & \gamma_7 \\ 1 & \gamma_2 & \gamma_4 & \gamma_5 \\ 0 & 1 & \gamma_2 & \gamma_3 \\ 0 & 0 & 1 & \gamma_1 \end{vmatrix}.$$

(d) **(5 marks)** Use part (c) and the theory of determinants to deduce that

$$c_7 = \begin{vmatrix} \binom{2}{0} & \binom{4}{0} & \binom{6}{0} & \binom{7}{0} \\ 1 & \binom{4}{2} & \binom{6}{2} & \binom{7}{2} \\ 0 & 1 & \binom{6}{4} & \binom{7}{4} \\ 0 & 0 & 1 & \binom{7}{6} \end{vmatrix}.$$

(e) **(5 marks)** Write down a generalisation of the expression in part (d) to $c_{2n-1}$. (The proof of this would be an immediate generalisation of the proof you have given for $c_7$.)

(f) **(5 marks)** Using a differential decomposition of the set of all permutations with pattern $(\nearrow\searrow)^\star$, and the set of all permutations with pattern $(\nearrow\searrow)^\star\nearrow$, prove that

$$\left[\frac{x^{2n-1}}{(2n-1)!}\right]\tan(x) = c_{2n-1}.$$

(15) **(Total: 10 marks)** We adopt the convention that every non-null sequence has a terminal maximal $<$-substring. For non-null sequences, let $f_i$ be an indeterminate marking a non-terminal maximal $<$-substring of length $i$ and let $g_i$ be an indeterminate marking a terminal maximal $<$-substring of length $i$, for $i = 1, 2, \ldots$. Let $f(x) = 1 + f_1x + f_2x^2 + \cdots$ and $g(x) = g_1x + g_2x^2 + \cdots$. (Note that the convention implies that $g$ has 0 as its constant term.)

(a) **(8 marks)** Prove that the generating series for the set of all non-null sequences with repect to the number of occurrences of the symbol $i$, marked by $x_i$, and the number of terminal and non-terminal maximal $<$-substrings of length $i$, marked by $f_i$ and $g_i$ respectively, is

$$\left(f^{-1} \circ \gamma\right)^{-1} \left(f^{-1} g \circ \gamma\right).$$

(b) **(2 marks)** Show how to deduce the Maximal Decomposition Theorem directly from the result stated in part (a).

(16) **(Total: 10 marks)** Let $h$ be an element of the pattern algebra and let

$$\mathsf{h} = \mathsf{q} - \mathsf{L}^t \mathsf{w} \mathsf{R}$$

be a left- or right-expansion of $\mathsf{h}$, where $\mathbf{L} = (\mathsf{L}_1, \ldots, \mathsf{L}_s)^t$, $\mathbf{R} = (\mathsf{R}_1, \ldots, \mathsf{R}_s)^t$ and $^t$ denotes transposition.

(a) **(5 marks)** Prove that

$$\mathbf{R}\mathsf{q}^{-1} = \mathbf{R}\mathsf{h}^{-1} - \mathbf{R}\mathsf{q}^{-1}\mathbf{L}^t\mathsf{w}\mathbf{R}\mathsf{h}^{-1}.$$

(b) **(5 marks)** Let $\left[\mathbf{R}\mathsf{h}^{-1}\right]_o$ be defined to be $\left(\left[\mathsf{R}_1\mathsf{h}^{-1}\right]_o, \ldots, \left[\mathsf{R}_s\mathsf{h}^{-1}\right]_o\right)$. Prove that the generating series $\left[\mathsf{h}^{-1}\right]_o$ is given in terms of $\left[\mathbf{R}\mathsf{h}^{-1}\right]_o$ by

$$\left[\mathsf{h}^{-1}\right]_o = \left[\mathsf{q}^{-1}\right]_o + \left[\mathsf{q}^{-1}\mathbf{L}^t\right]_o \left[\mathbf{R}\mathsf{h}^{-1}\right]_o,$$

where

$$\left[\mathbf{R}\mathsf{h}^{-1}\right]_o = \left(\mathbf{I}_s - \left[\mathbf{R}\mathsf{q}^{-1}\mathbf{L}^t\right]_o\right)^{-1} \left[\mathbf{R}\mathsf{q}^{-1}\right]_o.$$

(c) **(5 marks)** Find $\mathbf{R}$, $\mathbf{L}$ and $\mathsf{q}$ for $\mathsf{h} = 1 - \mathsf{u}^3\mathsf{d}^3$.

(17) **(Total: 10 marks)** Use the Maximal Decomposition Theorem to show the generating series for the set of all sequences with no identical symbols in adjacent positions is

$$\left(1-\sum_{i\geq 1}\frac{x_i}{1+x_i}\right)^{-1},$$

where $x_i$ marks the occurrence of the symbol $i$.

(18) An isolated symbol in a sequence is a symbol that is not adjacent to a copy of itself. Prove that the generating series for the set of all sequences with no isolated symbols is

$$\left(1-\sum_{i\geq 1}\left(\frac{x_i^3}{1+x_i^3}+\frac{x_i^2}{1+x_i^3}\right)\right)^{-1.}$$

**Comment:** You will need to use the $\gamma_k$'s for the relation "=". These are denoted by $\gamma_k^=$ and are defined by $\gamma_k^= = x_1^k + x_2^k + \cdots$.

(19) **(Total: 10 marks)** A sequence $\sigma_1\cdots\sigma_m$ has a *double fall* in position $i$ if $\sigma_{i-1}\geq\sigma_i\geq\sigma_{i+1}$. Let $c_n$ be the number of permutations of length $2n+1$ with no double falls in even positions. By using the result proved in Question 2(b), show that

$$\sum_{n\geq 0} c_n\frac{x^{2n+1}}{(2n+1)!}=\frac{\sin(x)}{1-x\sin(x)}.$$

(20) **(Total: 10 marks)** Let $T(x)$ be the (unique) solution of the functional equation

$$T = xe^T.$$

Find the power series expansion for $T^3(x)$.