C&O 330 - SOLUTIONS #4

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(1) **(15 points)** A *rise* in a sequence $\sigma_1 \cdots \sigma_p$ is a pair (σ_j, σ_{j+1}) such that $\sigma_j < \sigma_{j+1}$. Prove that the number of permutations on n symbols with exactly k rises is

$$\left[u^k \frac{x^n}{n!}\right] \frac{u-1}{u-e^{(u-1)x}}.$$

[**Hint:** Use the Maximal Decomposition Theorem and the Permutation Lemma.]

Solution: Let f_k mark the occurrence of a maximal $<$-substring of length k. Such a substring contains $k-1$ rises so

$$f(x) = 1 + \sum_{k \geq 1} u^{k-1} x^k = 1 + \frac{x}{1-ux} = \frac{1+(1-u)x}{1-x}.$$

Then, by the Maximal Decomposition Theorem, the generating series for sequences with respect to rises is

$$\begin{aligned} F &= \left(f^{-1} \circ \gamma^{<}\right)^{-1} \\ &= \left(\left(\frac{1-x}{1+(1-u)x}\right) \circ \gamma^{<}\right)^{-1} \\ &= \left(\sum_{i \geq 0} (u-1)^i \left(x^i - x^{i+1}\right) \circ \gamma^{<}\right)^{-1} \\ &= \left(\left(1 - \frac{1}{u-1} \sum_{i \geq 1} (u-1)^i x^i\right) \circ \gamma^{<}\right)^{-1} \\ &= \frac{u-1}{u - \sum_{i \geq 0} (u-1)^i \gamma_i^{<}}. \end{aligned}$$

Thus the generating series for permutations with respect to rises is, by the Permutation Lemma,

$$\Delta \frac{u-1}{u - \sum_{i \geq 0} (u-1)^i \gamma_i^{<}} = \frac{u-1}{u - \sum_{i \geq 0} (u-1)^i \frac{x^i}{i!}}$$

and the result follows.

(2) **(15 points)** Prove that the number of permutations of length n, with precisely k maximal $<$-substrings having length greater than or equal to 2,

is

$$\left[u^k \frac{x^n}{n!}\right] \left(\cosh(zx) - z^{-1} \sinh(zx)\right)^{-1}$$

where

$$z = \sqrt{1-u}.$$

[**Comment:** Recall that $\cosh(y) = \sum_{k \geq 0} \frac{y^{2k}}{(2k)!}$ and $\sinh(y) = \sum_{k \geq 0} \frac{y^{2k+1}}{(2k+1)!}$.]

Solution: Let u mark the occurrence of a maximal $<$-substring of length greater than or equal to 2. Then, following the argument in the previous problem,

$$\begin{aligned} f(x) &= 1 + x + u\left(x^2 + x^3 + \cdots\right) \\ &= 1 + x + \frac{ux^2}{1-x} \\ &= \frac{1 + (u-1)x^2}{1-x} \end{aligned}$$

so, by the Maximal Decomposition Theorem and the Permutation Lemma, the desired generating series is

$$\begin{aligned} F &= \Delta\left(\left(\frac{1-x}{1+(u-1)x^2}\right) \circ \gamma^{<}\right)^{-1} \\ &= \Delta\left(\left(\sum_{k\geq 0} (u-1)^k x^{2k} \circ \gamma^{<} - \sum_{k \geq 0} (u-1)^k x^{2k+1} \circ \gamma^{<}\right)^{-1}\right) \\ &= \left(\cosh(zx) - z^{-1}\sinh(zx)\right)^{-1} \end{aligned}$$

(3) **(15 points)** Let $D(x_1, x_2, \ldots)$ be the ordinary generating series for the number $d(k_1, \ldots, k_n)$ of sequences with k_i occurrences of i for $i = 1, \ldots, n$ such that adjacent symbols in the sequence are not equal.

(a) **(6 points)** Find the generating series F for the set of all sequences over $\mathcal{N}_n = \{1, \ldots, n\}$ with respect to the number of symbols of each type, where x_i marks the occurrence of the symbol i, for $i = 1, \ldots, n$.
Solution: F is the generating series for $\{1. \ldots, n\}^{\star}$, which is

$$\frac{1}{1 - \sum_{i=1}^{n} x_i}.$$

(b) **(9 points)** Find D by first determining how to construct each string over $\mathcal{N}_n$ from a unique sequence counted by $d(k_1, \ldots, k_n)$.
Solution: The set $\{1. \ldots, n\}^{\star}$ can be constructed by selecting a sequence σ in it, and replacing each symbol i by $i\{i\}^{\star}$ for each i, and for each σ. Thus, at the level of generating series,

$$D(x_1, \ldots, x_n)\,|_{x_i \mapsto x_i(1-x_i)^{-1}, i=1,2,\ldots} = F = \frac{1}{1 - \sum_{i=1}^{n} x_i}$$

from the previous part, whence

$$D\left(\frac{x_1}{1-x_1}, \ldots, \frac{x_n}{1-x_n}\right) = \frac{1}{1 - \sum_{i=1}^{n} x_i}.$$

Let
$$y_i = \frac{x_i}{1-x_i}.$$
Then
$$x_i = \frac{y_i}{1+y_i}$$
so
$$D\left(y_1,\ldots,y_n\right) = \frac{1}{1-\sum_{i=1}^{n}\frac{y_i}{1+y_i}}$$

(4) **(15 points)** Let $D\left(x_1,x_2,\ldots\right)$ be the ordinary generating series for the number $d\left(k_1,\ldots,k_n\right)$ of sequences with k_i occurrences of i for $i=1,\ldots,n$ such that adjacent symbols in the sequence are not equal.

(a) **(7 points)** Use the Maximal Decomposition Theorem to prove that
$$d\left(k_1,\ldots,k_n\right) = \left[x_1^{k_1}\cdots x_n^{k_n}\right]\left(1-\sum_{i\geq 1}x_i\left(1+x_i\right)^{-1}\right)^{-1}.$$
[**Comment:** This question is the same as the previous one. However. this time you are asked to use the maximal Decomposition Theorem.]
Solution: Let $\pi_1 = " = "$. Then $f(x) = 1+x$. Then by the Maximal Decomposition Theorem, the required generating series is
$$\begin{aligned}
F &= \left(\left(1+x\right)^{-1}\circ\gamma^{=}\right)^{-1} \\
&= \left(1+\sum_{i\geq 1}\left(-1\right)^i x^i\circ\gamma^{=}\right)^{-1} \\
&= \left(1+\sum_{i\geq 1}\left(-1\right)^i \gamma_i^{=}\right)^{-1} \\
&= \left(1+\sum_{i\geq 1}\sum_{k\geq 1}\left(-1\right)^i x_k^i\right)^{-1} \\
&= \left(1-\sum_{k\geq 1}x_k\left(1+x_k\right)^{-1}\right)^{-1}.
\end{aligned}$$

(b) **(8 points)** Let $u \leftrightarrow <$ and $d \leftrightarrow \geq$. By using part (a), or otherwise, state a combinatorial interpretation of
$$\phi\left(D\left(d,ud,u^2d,u^3d,\ldots\right)\right).$$
Solution: This is the generating series for the number of sequences such that no pairof adjacent maximal $<$-substrings have the same length.

(5) **(25 points)** Let $u \leftrightarrow <$ and $d \leftrightarrow \geq$, and let $x_1,x_2,\ldots$ be commuting indeterminates, and let ϕ be the partial homomorphism associated with the Pattern Algebra.

(a) **(5 points)** Prove that

$$\phi\left((ud)^3 u\right) = \phi\left((ud)^2 u\right)\gamma_2 - \phi\left((ud)\,u\right)\gamma_4 + \phi(u)\,\gamma_6 - \gamma_8.$$

Solution:

$$\begin{aligned}(ud)^3 u &= (ud)^2 (ud)\, u \\ &= (ud)^2 u\,(w-u)\,u \\ &= (ud)^2 uwu - (ud)^2 u^3 \\ &= (ud)^2 uwu - (ud)\,u\,(w-u)\,u^3 \\ &= (ud)^2 uwu - (ud)\,uwu^3 + (ud)\,u^5 \\ &= (ud)^2 uwu - (ud)\,uwu^3 + uwu^5 - u^7,\end{aligned}$$

so, applying the partial homomorphism ϕ, we obtain the result.

(b) **(5 points)**By deducing similar expressions for $\phi\left((ud)^2 u\right)$ and $\phi\left((ud)\,u\right)$, prove that these expressions satisfy the system of simultaneous equations

$$\begin{bmatrix} 1 & -\gamma_2 & \gamma_4 & -\gamma_6 \\ 0 & 1 & -\gamma_2 & \gamma_4 \\ 0 & 0 & 1 & -\gamma_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi\left((ud)^3 u\right) \\ \phi\left((ud)^2 u\right) \\ \phi\left((ud)\,u\right) \\ \phi(u) \end{bmatrix} = \begin{bmatrix} -\gamma_8 \\ \gamma_6 \\ -\gamma_4 \\ \gamma_2 \end{bmatrix}.$$

Solution: In similar fashion,

$$\begin{aligned}(ud)^2 u &= (ud)\,uwu - (ud)\,u^3 \\ &= (ud)\,uwu - uwu^3 + u^5,\end{aligned}$$

so

$$\phi\left((ud)^2 u\right) = \phi\left((ud)\,u\right)\gamma_2 - \phi(u)\,\gamma_4 + \gamma_6,$$

and

$$(ud)\,u = uwu - u^3,$$

so

$$\phi\left((ud)\,u\right) = \phi(u)\,\gamma_2 - \gamma_4.$$

Finally, $\phi(u) = \gamma_2$. The result now follows by presenting these four equations matricially.

(c) **(10 points)** By solving this equation for $\phi\left((ud)^3 u\right)$, or otherwise, prove that

$$\left[\frac{x^8}{8!}\right]\sec(x) = \begin{vmatrix} \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \binom{8}{8} \\ 1 & \binom{6}{2} & \binom{6}{4} & \binom{6}{6} \\ 0 & 1 & \binom{4}{2} & \binom{4}{4} \\ 0 & 0 & 1 & \binom{2}{2} \end{vmatrix}.$$

Solution: By Crámer's Rule

$$\phi\left((ud)^3 u\right) = \begin{vmatrix} -\gamma_8 & -\gamma_2 & \gamma_4 & -\gamma_6 \\ \gamma_6 & 1 & -\gamma_2 & \gamma_4 \\ -\gamma_4 & 0 & 1 & -\gamma_2 \\ \gamma_2 & 0 & 0 & 1 \end{vmatrix}$$

since the coefficient matrix is unidiagonal. Interchange the order of the columns and use elementary row and column operations to obtain

$$\phi\left((ud)^3 u\right) = \begin{vmatrix} \gamma_2 & \gamma_4 & \gamma_6 & \gamma_8 \\ 1 & \gamma_2 & \gamma_4 & \gamma_6 \\ 0 & 1 & \gamma_2 & \gamma_4 \\ 0 & 0 & 1 & \gamma_2 \end{vmatrix}.$$

Then, applying the permutation homomorphism we have

$$\Delta\phi\left((ud)^3 u\right) = \begin{vmatrix} \frac{x^2}{2!} & \frac{x^4}{4!} & \frac{x^6}{6!} & \frac{x^8}{8!} \\ 1 & \frac{x^2}{2!} & \frac{x^4}{4!} & \frac{x^6}{6!} \\ 0 & 1 & \frac{x^2}{2!} & \frac{x^4}{4!} \\ 0 & 0 & 1 & \frac{x^2}{2!} \end{vmatrix}.$$

Now multiply columns 1,2 and 3 by x^6, x^4 and x^2, respectively, and divide rows 2, 3 and 4 by x^6, x^4 and x^2, respectively, to obtain,

$$\Delta\phi\left((ud)^3 u\right) = x^8 \begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} \\ 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} \\ 0 & 1 & \frac{1}{2!} & \frac{1}{4!} \\ 0 & 0 & 1 & \frac{1}{2!} \end{vmatrix}.$$

Now divide columns 1,2 and 3 by $6!, 4!$ and $2!$, respectively, and multiply rows 2, 3 and 4 by $6!, 4!$ and $2!$, respectively, to obtain,

$$\Delta\phi\left((ud)^3 u\right) = \frac{x^8}{8!} \begin{vmatrix} \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \binom{8}{8} \\ 1 & \binom{6}{2} & \binom{6}{4} & \binom{6}{6} \\ 0 & 1 & \binom{4}{2} & \binom{4}{4} \\ 0 & 0 & 1 & \binom{2}{2} \end{vmatrix}.$$

But the left hand side is the generating series for the number of <-alternating permutations of length 8. The result follows.

(d) **(5 points)** Try to see how an expression of this sort may be deduced for

$$\left[\frac{x^{2n}}{(2n)!}\right] \sec(x),$$

or conjecture what this might be.