C&O 330 - SOLUTIONS #3

PROFESSOR D.M. JACKSON

(1) **(15 points)** Let K_n denote the complete graph on n vertices.

(a) **(7 points)** Show that the number of ways of covering K_n with paths of vertex-length 1 or more is

$$\left[\frac{x^n}{n!}\right] \exp\left(\frac{x(2-x)}{2(1-x)}\right).$$

Solution: Let $\mathcal{S}$ be the set of all such coverings of K_n for $n = 0, 1, 2, \ldots,$ and let $\mathcal{L}$ be the set of all non-null undirected labelled paths. Then

$$\mathcal{S} \leftrightarrow \mathcal{U} \circledast \mathcal{L}.$$

Let $\mathcal{D}$ be the set of all non-null directed labelled paths. Then, with the exception of the path of (vertex) length 1, any path in $\mathcal{L}$ may be obtained from precisely 2 paths in $\mathcal{D}$ by removing the orientation of the paths. Thus there is a natural action of $\mathfrak{S}_2$ on $\mathcal{D}$ to give $\mathcal{L}$, where $\mathfrak{S}_2$ (the symmetric group on 2 sumbols) acts neutrally on a path of length 1. Then

$$\begin{aligned} \mathcal{L} &\leftrightarrow \mathcal{D}/\mathfrak{S}_2 \\ &\leftrightarrow \mathcal{P}/\mathfrak{S}_2 \end{aligned}$$

where $\mathcal{P}$ is the set of all permutations on $\{1, \ldots, n\}$ for all $n \geq 0$. Let $\omega(c)$ be the number of labels in $c \in \mathcal{S}$. Then

$$\mathcal{S} \leftrightarrow \mathcal{U} \circledast (\mathcal{P}/\mathfrak{S}_2)$$

is additively ω-preserving. Let $S(x) = [(\mathcal{S}, \omega)]_e(x)$. By the $\circledast$-Product Lemma

$$S = \exp\left([(\mathcal{P}/\mathfrak{S}_2, \omega)]_e(x)\right).$$

But $[(\mathcal{P}, \omega)]_e(x) = \sum_{k\geq 1} x^k$, so

$$[(\mathcal{P}/\mathfrak{S}_2, \omega)]_e(x) = x + \frac{1}{2}\sum_{k\geq 2} x^k = \frac{x}{2}\frac{2-x}{1-x},$$

recalling that $\mathfrak{S}_2$ acts neutrally on paths of length 1. Thus the desired number is

$$\left[\frac{x^n}{n!}\right] S(x) = \left[\frac{x^n}{n!}\right] \exp\left(\frac{x}{2}\frac{2-x}{1-x}\right).$$

(b) **(8 points)** Show that the number of ways of covering K_n with cycles of length 3 or more is

$$\left[\frac{x^n}{n!}\right] (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{4}\right).$$

Solution: The method is the same. Let $\mathcal{V}$ be the set of all such covers, and let $\mathcal{C}_{\geq 3}$ be the set of all labelled cycles of length greater than or equal to 3. Let $\omega(v)$ be the number of labels in $v \in \mathcal{V}$. Then

$$\mathcal{V} \leftrightarrow \mathcal{U} \circledast (\mathcal{C}_{\geq 3}/\mathfrak{S}_2).$$

This is clearly additively ω-preserving. Let $V(x) = [(\mathcal{V}, \omega)]_e(x)$. Then, by the $\circledast$-Product Lemma,

$$\mathcal{V} = \exp\left([(\mathcal{C}_{\geq 3}/\mathfrak{S}_2, \omega)]_e(x)\right)$$

By the Disjoint Cycle Decomposition

$$\mathcal{P} \leftrightarrow \mathcal{U} \circledast \mathcal{C},$$

where $\mathcal{C}$ is the set of all non-null labelled cycles and $\mathcal{P}$ is the set of all permutations. Let $C(x) = [(\mathcal{C}, \omega)]_e(x)$. Then $(1-x)^{-1} = \exp(C)$ so $C = \log\left((1-x)^{-1}\right) = \sum_{k\geq 1} x^k/k$. Then

$$[(\mathcal{C}_{\geq 3}/\mathfrak{S}_2, \omega)]_e(x) = \frac{1}{2}\left(\log\left((1-x)^{-1}\right) - x - \frac{x^2}{2}\right).$$

It follows that

$$\begin{aligned} S(x) &= \exp\left(\frac{1}{2}\left(\log\left((1-x)^{-1}\right) - x - \frac{x^2}{2}\right)\right) \\ &= \exp\left(\log\left((1-x)^{-1/2}\right) - \frac{x}{2} - \frac{x^2}{4}\right), \end{aligned}$$

giving the result.

(2) **(15 points)** A simple graph is a graph with no loops or multiple edges.

(a) **(7 points)** Show that the number of simple connected labelled graphs on n vertices and i edges is

$$\left[y^i \frac{x^n}{n!}\right] \log\left(\sum_{m\geq 0} \frac{x^m}{m!}(1+y)^{\binom{m}{2}}\right).$$

Solution: Let

$$\binom{\mathcal{N}_m}{2}$$

denote the set of all unordered pairs of distinct symbols from $\{1, \ldots, m\}$. Now any simple graph on $\mathcal{N}_m = \{1, \ldots, m\}$ can be constructed by selecting each pair of vertices, and either drawing an edge between them (indicated by 1) or not drawing an edge between them (indicated by 1). This can be encoded as a function $f \colon \binom{\mathcal{N}_m}{2} \to \{0,1\}$. Let $\mathcal{G}$ be the set of all simple labelled graphs. Then

$$\mathcal{G} \leftrightarrow \cup_{m\geq 0} \{0,1\}^{\binom{\mathcal{N}_m}{2}} \times \mathcal{N}_m.$$

Let $\mathcal{H}$ be the set of all non-null connected labelled simple graphs. Then each graph in $\mathcal{G}$ can be expressed uniquely as a union of connected graphs from $\mathcal{H}$ with suitable relabelling. Thus

$$\mathcal{G} \leftrightarrow \mathcal{U} \circledast \mathcal{H}.$$

For a graph g, let $\lambda(g)$ be the number of vertices of g and μ the number of edges of g. Clearly, both bijections are additively $\lambda\otimes\mu$-preserving. Let x be an exponential indeterminate marking vertices (they are labelled), and let y be an ordinary indeterminate marking edges (they are not labelled). Let $G(x;y) = \left[(\mathcal{G},\lambda\otimes\mu)\right]_{e,o}(x;y)$ and $H(x;y) = \left[(\mathcal{H},\lambda\otimes\mu)\right]_{e,o}(x;y)$. Then, from the second decomposition,

$$G = \exp(H)$$

so

$$H = \log(G).$$

Also, from the first decomposition,

$$\begin{aligned} G &= \sum_{m\geq 0}\left[\left(\{0,1\}^{\binom{\mathcal{N}_m}{2}},\mu\right)\right]_o(y)\cdot\left[(\mathcal{N}_m,\lambda)\right]_e(x) \\ &= \sum_{m\geq 0}(1+y)^{\binom{m}{2}}\frac{x^m}{m!}, \end{aligned}$$

so

$$H = \log\left(\sum_{m\geq 0}(1+y)^{\binom{m}{2}}\frac{x^m}{m!}\right)$$

and the result follows.

(b) **(8 points)** Show that the number of simple labelled graphs with k components on n vertices and i edges is

$$\left[y^i\frac{x^n}{n!}\right]\frac{1}{k!}\left(\log\left(\sum_{m\geq 0}\frac{x^m}{m!}(1+y)^{\binom{m}{2}}\right)\right)^k$$

Solution: Let $\mathcal{K}_k$ be the set of all labelled simple graphs with k components. Then

$$\mathcal{K}_k \leftrightarrow \{1,\ldots,k\}\circledast\mathcal{H}.$$

Let $K_k(x;y) = \left[(\mathcal{K}_k,\lambda\otimes\mu)\right]_{e,o}(x;y)$. Then, by the $\circledast$-Lemma, we have

$$K_k = \frac{1}{k!}H^k.$$

The result follows from the first part.

(3) **(15 points)** Show that the number of $\{0,1\}$-matrices with m rows, exactly k of which are empty (contain only 0s), n columns, exactly l of which are empty, and containing p 1s is

$$(-1)^{m+n+k+l}\sum_{i,j\geq 0}(-1)^{i+j}\binom{m}{i}\binom{n}{j}\binom{m-i}{k}\binom{n-j}{l}\binom{ij}{p}.$$

Solution: Use the filter decomposition from class and refine the weight function by tensoring it up twice.

(4) **(15 points)** Consider n lines in general position, so that no three are concurrent. A *frame* consists of n of the $\binom{n}{2}$ points of intersection, such that no three of the n points lie on the same line. Show that the number of frames on n lines is

$$\left[\frac{x^n}{n!}\right] (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{4}\right).$$

Solution: Take the dual of this problem. This is: there are n points in general position in the plane. A *dual frame* consists of n of the $\binom{n}{2}$ lines joining pairs of points, such that no three of the n lines lie meet in the same point. Thus a dual frame consists of cycles of length greater than or equal to 3 that cover the n points. This problem hasa been solved in 1(b). The result therefore follows.

(5) **(15 points)** Show that the number of sequences on $\{1,\ldots,n\}$ of length m and containing k different objects is

$$\binom{n}{k}\left[\frac{x^m}{m!}\right](e^x-1)^k.$$

Solution: Such a sequence can be encoded by an ordered partition of $\{1,\ldots,m\}$ into k subsets such that the i-th subset contains the positions in the sequence at which the symbol i occurs. Since k different symbols occur, these subsets must be non-empty. Let $\mathcal{S}_k$ be the set of all sequences in $\mathcal{N}_k$ in which each symbol occurs at least once. Then, by the above comment,

$$\mathcal{S}_k \leftrightarrow (\mathcal{U}-\varepsilon)^{\star k}.$$

Then the number of sequences of length n in $\mathcal{S}_k$ is $\left[\frac{x^m}{m!}\right](e^x-1)^k$. But the k symbols can be chosen from $\{1,\ldots,n\}$ in $\binom{n}{k}$ ways. Thus the required number of sequences is

$$\binom{n}{k}\left[\frac{x^m}{m!}\right](e^x-1)^k.$$