C&O 330 - ASSIGNMENT #4

DUE FRIDAY, 19 NOVEMBER AT 10:31PM

(1) **(20 points)**
   (a) **(15 points)** Let $\mathsf{u} \leftrightarrow <$ and $\mathsf{d} \leftrightarrow \geq$. Let $a_1, \ldots, a_4$ be positive integers. Find the number of permutations with pattern
   $$\mathsf{u}^{a_1-1}\mathsf{d}\mathsf{u}^{a_2-1}\mathsf{d}\mathsf{u}^{a_3-1}\mathsf{d}\mathsf{u}^{a_4-1},$$
   expressing your result as a determinant of binomial numbers.
   (b) **(5 points)** On the basis of this evidence state a conjecture of the number of permutations with pattern
   $$\mathsf{u}^{a_1-1}\mathsf{d}\mathsf{u}^{a_2-1}\mathsf{d}\cdots\mathsf{u}^{a_m-1}\mathsf{d}\mathsf{u}^{a_{m+1}-1}.$$

(2) **(20 points)** A *dodecahedron* is a regular solid consisting of 12 regular pentagons arranged so that each vertex of the dodecahedron is incident with 3 pentagons and that every edge of the dodecahedron is incident with 2 pentagons.
   (a) **(15 points)** Find the cycle index polynomial for the automorphism group of the dodecahedron acting on the 12 faces of the dodecahedron.
   (b) **(5 points)** Find the generating series for the number of ways of painting the faces of the dodecahedron with two colours.
   (c) **(5 points)** Confirm the coefficient giving the number of ways of painting the faces of the dodecahedron with 2 red faces and 10 green faces by generating the paintings combinatorially.

(3) **(20 points)** The following question is quite a long exercise in the pattern algebra and it will take time.
   (a) **(10 points)** Find the generating series F for the set of all permutations with pattern $\left\{<^2\geq^2\right\}^\star$.
   (b) **(10 points)** Use a series of differential constructions to obtain a system of simultaneous differential equations for F.
   [**Hint:** Delete the largest element of such a permutation, which then decomposes into two constituents, one of which may be null. Characterise the patterns of the two constituents. Repeat the differential process on the two new sets, which then introduces further sets. Ultimately you will observe than no new sets are introduced and that you therefore have a closed system of equations. That is to say, the process is naturally terminating.]
   [**Comment:** The generating series F determined in part (a) from the pattern algebra is a solution of this system of equations. Such systems of equations are called *matrix Ricatti* equations.]