C&O 330 - ASSIGNMENT #4

DUE FRIDAY, 5 NOVEMBER AT 10:31AM

(1) **(15 points)** A *rise* in a sequence $\sigma_1 \cdots \sigma_p$ is a pair (σ_j, σ_{j+1}) such that $\sigma_j < \sigma_{j+1}$. Prove that the number of permutations on n symbols with exactly k rises is

$$\left[u^k \frac{x^n}{n!}\right] \frac{u-1}{u-e^{(u-1)x}}.$$

[**Hint:** Use the Maximal Decomposition Theorem and the Permutation Lemma.]

(2) **(15 points)** Prove that the number of permutations of length n, with precisely k maximal $<$-substrings having length greater than or equal to 2, is

$$\left[u^k \frac{x^n}{n!}\right] \left(\cosh(zx) - z^{-1}\sinh(zx)\right)^{-1}$$

where

$$z = \sqrt{1-u}.$$

[**Comment:** Recall that $\cosh(y) = \sum_{k\geq 0} \frac{y^{2k}}{(2k)!}$ and $\sinh(y) = \sum_{k\geq 0} \frac{y^{2k+1}}{(2k+1)!}$.]

(3) **(15 points)** Let $D(x_1, x_2, \ldots)$ be the ordinary generating series for the number $d(k_1, \ldots, k_n)$ of sequences with k_i occurrences of i for $i = 1, \ldots, n$ such that adjacent symbols in the sequence are not equal.
   (a) **(6 points)** Find the generating series F for the set of all sequences over $\mathcal{N}_n = \{1, \ldots, n\}$ with respect to the number of symbols of each type, where x_i marks the occurrence of the symbol i, for $i = 1, \ldots, n$.
   (b) **(9 points)** Find D by first determining how to construct each string over $\mathcal{N}_n$ from a unique sequence counted by $d(k_1, \ldots, k_n)$.

(4) **(15 points)** Let $D(x_1, x_2, \ldots)$ be the ordinary generating series for the number $d(k_1, \ldots, k_n)$ of sequences with k_i occurrences of i for $i = 1, \ldots, n$ such that adjacent symbols in the sequence are not equal.
   (a) **(7 points)** Use the Maximal Decomposition Theorem to prove that

$$d(k_1, \ldots, k_n) = \left[x_1^{k_1} \cdots x_n^{k_n}\right] \left(1 - \sum_{i\geq 1} x_i (1+x_i)^{-1}\right)^{-1}.$$

   [**Comment:** This question is the same as the previous one. However. this time you are asked to use the maximal Decomposition Theorem.]
   (b) **(8 points)** Let $u \leftrightarrow <$ and $d \leftrightarrow \geq$. By using part (a), or otherwise, state a combinatorial interpretation of

$$\phi\left(D\left(d, ud, u^2d, u^3d, \ldots\right)\right).$$

(5) **(25 points)** Let $u \leftrightarrow <$ and $d \leftrightarrow \geq$, and let $x_1, x_2, \ldots$ be commuting indeterminates, and let ϕ be the partial homomorphism associated with the Pattern Algebra.

(a) **(5 points)** Prove that

$$\phi\left((ud)^3 u\right) = \phi\left((ud)^2 u\right)\gamma_2 - \phi\left((ud)\, u\right)\gamma_4 - \phi\left(u\right)\gamma_6 + \gamma_8.$$

(b) **(5 points)** By deducing similar expressions for $\phi\left((ud)^2 u\right)$ and $\phi\left((ud)\, u\right)$, prove that these expressions satsify the system of simultaneous equations

$$\begin{bmatrix} 1 & -\gamma_2 & \gamma_4 & -\gamma_6 \\ 0 & 1 & -\gamma_2 & \gamma_4 \\ 0 & 0 & 1 & -\gamma_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi\left((ud)^3 u\right) \\ \phi\left((ud)^2 u\right) \\ \phi\left((ud)\, u\right) \\ \phi\left(u\right) \end{bmatrix} = \begin{bmatrix} -\gamma_8 \\ \gamma_6 \\ -\gamma_4 \\ \gamma_2 \end{bmatrix}.$$

(c) **(10 points)** By solving this equation for $\phi\left((ud)^3 u\right)$, or otherwise, prove that

$$\left[\frac{x^8}{8!}\right]\sec\left(x\right) = \begin{vmatrix} \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \binom{8}{8} \\ 1 & \binom{6}{2} & \binom{6}{4} & \binom{6}{6} \\ 0 & 1 & \binom{4}{2} & \binom{4}{4} \\ 0 & 0 & 1 & \binom{2}{2} \end{vmatrix}.$$

(d) **(5 points)** Try to see how an expression of this sort may be deduced for

$$\left[\frac{x^{2n}}{(2n)!}\right]\sec\left(x\right),$$

or conjecture what this might be.