C&O 330 - ASSIGNMENT #3

DUE FRIDAY, 22 OCTOBER AT 10:31PM

(1) **(15 points)** Let K_n denote the complete graph on n vertices.

(a) **(7 points)** Show that the number of ways of covering K_n with paths of vertex-length 1 or more is

$$\left[\frac{x^n}{n!}\right] \exp\left(\frac{x\,(2-x)}{2\,(1-x)}\right).$$

(b) **(8 points)** Show that the number of ways of covering K_n with cycles of length 3 or more is

$$\left[\frac{x^n}{n!}\right] (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{4}\right)$$

(2) **(15 points)** A simple graph is a graph with no loops or multiple edges.

(a) **(7 points)** Show that the number of simple connected labelled graphs on n vertices and i edges is

$$\left[y^i \frac{x^n}{n!}\right] \log\left(\sum_{m\geq 0} \frac{x^m}{m!}\,(1+y)^{\binom{m}{2}}\right).$$

(b) **(8 points)** Show that the number of simpled labelled graphs with k components on n vertices and i edges is

$$\left[y^i \frac{x^n}{n!}\right] \frac{1}{k!} \left(\log\left(\sum_{m\geq 0} \frac{x^m}{m!}\,(1+y)^{\binom{m}{2}}\right)\right)^k$$

(3) **(15 points)** Show that the number of $\{0,1\}$-matrices with m rows, exactly k of which are empty (contain only 0s), n columns, exactly l of which are empty, and containing p 1s is

$$(-1)^{m+n+k+l} \sum_{i,j\geq 0} (-1)^{i+j} \binom{m}{i}\binom{n}{j}\binom{m-i}{k}\binom{n-j}{l}\binom{ij}{p}.$$

(4) **(15 points)** Consider n lines in general position, so that no three are concurrent. A *frame* consists of n of the $\binom{n}{2}$ points of intersection, such that no three of the n points lie on the same line. Show that the number of frames on n lines is

$$\left[\frac{x^n}{n!}\right] (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{2}\right).$$

(5) **(15 points)** Show that the number of sequences on $\{1,\ldots,n\}$ of length m and containing k different objects is

$$\binom{n}{k}\left[\frac{x^m}{m!}\right](e^x-1)^k\,.$$