VOJTA'S CONJECTURE IMPLIES THE BATYREV-MANIN CONJECTURE FOR $K3$ SURFACES

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Abstract. Vojta's Conjectures are well known to imply a wide range of results, known and unknown, in arithmetic geometry. In this paper, we add to the list by proving that they imply that rational points tend to repel each other on algebraic varieties with nonnegative Kodaira dimension. We use this to deduce, from Vojta's Conjectures, conjectures of Batyrev-Manin and Manin on the distribution of rational points on algebraic varieties. In particular, we show that Vojta's Main Conjecture implies the Batyrev-Manin Conjecture for $K3$ surfaces.

1. Introduction

In 1987, in [16], Vojta made a series of wide-ranging and deep conjectures about the distribution of rational points on algebraic varieties. They imply the Masser-Oesterlé *abc* conjecture, the Bombieri-Lang Conjecture that the set of rational points on a variety of general type is not Zariski dense, and a host of other well-known conjectures in number theory. (For more details on the various implications of Vojta's conjectures, see [16] or [4], section F.5.3.)

Vojta's Main Conjecture is known in only a few special cases, although some of these special cases are extremely significant. It is known for curves, being the union of Roth's Theorem for rational curves ([11]), Siegel's Theorem for elliptic curves ([12]), and Faltings' Theorem for curves of general type ([2]). It is also known for abelian varieties, by another theorem of Faltings ([3]). The case $X = \mathbb{P}^n$ and D a union of hyperplanes is also known, due originally to Schmidt for archimedean places S ([15]), and Schlickewei for a general set of places ([14]), and is the famous Schmidt Subspace Theorem. For a precise statement of Vojta's Main Conjecture, see section 2.

For varieties with a canonical class of which some multiple is effective (that is, for varieties with nonnegative Kodaira dimension), there

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is a lower bound for a height with respect to the canonical class outside a proper Zariski closed subset. In this case, Vojta's conjectures imply that rational points repel each other, in the sense that two rational points that are close to one another (with respect to, say, some archimedean metric) must have large height relative to this distance. This idea is made precise by the Repulsion Principle, Theorem 3.1, in section 3.

From the Repulsion Principle, one can easily deduce that on an algebraic variety of nonnegative Kodaira dimension, rational points must be quite sparse. For $K3$ surfaces, this is precisely the famous conjecture of Batyrev and Manin ([1]), and in section 4 we deduce this conjecture from Vojta's Main Conjecture, as a corollary of a more general result (Theorem 4.2). Finally, in section 5, we deduce Manin's Rational Curve Conjecture ([7]) from the Repulsion Principle.

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2. Vojta's Main Conjecture

Vojta's Main Conjecture requires a lot of terminology to state, so we list some of it here for convenience.

k	A number field
S	A finite set of places of k
X	A smooth, projective, algebraic variety defined over k
K	The canonical divisor class of X
D	A normal crossings divisor on X (see Definition 2.3)
L	A big divisor on X (see Definition 2.2)
h_K, h_L	Logarithmic height functions with respect to K and L
$h_{D,v}$	A local height function for D with respect to a place v of k (see Definition 2.4)
$m_S(D,\cdot)$	A proximity function for D with respect to S, given by $m_S(D,P)=\sum_{v\in S} h_{D,v}(P)$

The conjecture we are primarily interested in here is the following (see Conjecture 3.4.3 from [16]):

Conjecture 2.1 (Vojta's Main Conjecture). *Choose any $\epsilon > 0$. Then there exists a nonempty Zariski open set $U = U(\epsilon) \subset X$ such that for every k-rational point $P \in U(k)$, we have the following inequality:*

$$m_S(D,P) + h_K(P) \le \epsilon h_L(P) \tag{1}$$

For a general discussion of this conjecture, and an explanation of all the terminology, we refer the reader to [16]. However, of particular interest may be the stronger conjecture that the exceptional set Z can be chosen to be the same for all number fields k. Note, however, that this uniform Z may not be minimal for all number fields k, and that the smallest number field for which the uniform Z is minimal may increase arbitrarily with shrinking ϵ.

Further discussions of arithmetic distance functions and heights can be found in, for example, [13]. However, for the reader's convenience, we recall the definitions of some of the less common terms mentioned in Conjecture 2.1:

Definition 2.2. *A divisor D on a smooth algebraic variety X is* **big** *if and only if*

$$\textit{lim inf}_{m\to\infty}\frac{h^0(X,mD)}{m^n} > 0$$

where $n = \dim X$.

By a theorem of Kodaira (appendix of [5]), a divisor on a smooth, projective variety X is big if and only if it can be written as the sum $D = A + E$ of an ample divisor A and an effective divisor E. Note that in particular every ample divisor is big.

Definition 2.3. *Let X be a smooth algebraic variety defined over the complex numbers. A divisor D on X has* **normal crossings** *if and only if it is effective, has no multiple components, and for every point P in the support of D, there are analytic functions $z_1, \ldots, z_n$ on an analytic neighbourhood U of P such that $D \cap U$ is the locus $z_1 \ldots z_n = 0 \cap U$, where $n \leq \dim X$.*

For a smooth algebraic variety X defined over a number field k, we say that a divisor D has **normal crossings** *if and only if for every embedding $k \hookrightarrow \mathbb{C}$, the corresponding divisor $D_{\mathbb{C}}$ on $X_{\mathbb{C}}$ has normal crossings. (It is easy to see that if $D_{\mathbb{C}}$ has normal crossings for any one embedding of k in $\mathbb{C}$, then $D_{\mathbb{C}}$ will have normal crossings for all such embeddings.)*

Definition 2.4. *Let X be a smooth, projective, algebraic variety defined over a number field k, let D be a divisor on X, and let v be a place of k. Let k_v be the completion of k with respect to v, and let $\overline{k_v}$ be an algebraic closure of k_v. Let $\|\cdot\|_v$ be the absolute value corresponding to v, extended to an absolute value on $\overline{k_v}$. A* **local height function** *for D at v is a function $h_{D,v}\colon X(\overline{k_v}) - Supp(D) \to \mathbb{R}$ such that for all P in $X(\overline{k_v})$, there is a Zariski open neighbourhood U of P such that D is represented locally by a rational function f on U and:*

$$h_{D,v}(P) = -\frac{1}{[k:\mathbb{Q}]}\log\|f(P)\|_v + \alpha(P)$$

where α is a continuous function on $U(\overline{k_v})$.

If D is an effective cycle on X that can be written as the scheme-theoretic intersection of a finite number of effective divisors $D = \bigcap_i D_i$, then we define $h_{D,v} = \min_i h_{D_i,v}$. Note that $h_{D,v}$ can be defined on $X(\overline{k_v}) - Supp(D)$ by setting $h_{D_i,v}(P) = \infty$ if $P \in Supp(D_i)$.

It is not hard to see that if one chooses a different set of divisors $\{D_i\}$, then the resulting local height function differs from the original by a bounded function.

Intuitively, one may think of the local height functions as satisfying $h_{D,v}(P) = -\log \operatorname{dist}_v(P, D)$, where $\operatorname{dist}_v(P, D)$ denotes the distance from P to the support of D. Note that in [16], local height functions are called Weil functions.

In what follows, we will want to apply Conjecture 2.1 to the slightly more general case in which D is a cycle of arbitrary codimension. To do this, we return to and slightly generalise the notation from Vojta's Main Conjecture (Conjecture 2.1):

D'	A cycle on X that is contained in a normal crossings divisor D
$h_{D',v}$	A local height function for D' with respect to v
$m_S(D', \cdot)$	A proximity function for D' with respect to S, given by $m_S(D', P) = \sum_{v \in S} h_{D',v}(P)$

Conjecture 2.5 (Vojta's Main Conjecture for general cycles)**.** *Choose any $\epsilon > 0$. Then there exists a nonempty Zariski open set $U = U(\epsilon) \subset X$ such that*

$$m_S(D', P) + h_K(P) \le \epsilon h_L(P) \tag{2}$$

for every k-rational point $P \in U(k)$.

This follows immediately from Vojta's Main Conjecture and the observation that $m_S(D', P) \le m_S(D, P)$. As with Vojta's Main Conjecture, it has been conjectured that the exceptional set Z in Conjecture 2.5 can be chosen to be the same for any number field k.

3. Repulsion of rational points for varieties of nonnegative Kodaira dimension

This section contains a technical result which contains the heart of the proofs of the main results of the paper. In particular, Theorem 3.1 implies that the rational points of low height on a variety of nonnegative Kodaira dimension (such as a $K3$ surface) should be distributed very sparsely away from subvarieties of negative Kodaira dimension (such as rational curves).

A key observation is that the v-adic distance from P to Q on X is, up to multiplication by a bounded nonvanishing function, the same as the v-adic distance from (P, Q) to the diagonal D on $X \times X$. The latter quantity is related to the proximity function in Vojta's conjectures by $m_{D,v}(P, Q) = -\log \operatorname{dist}_v(P, Q)$.

Theorem 3.1 (Repulsion Principle)**.** *Let X be any smooth, projective variety of nonnegative Kodaira dimension defined over a number field k. For every smooth, projective variety V birational to a subvariety of $X \times X$, make the following two assumptions:*

1. *If V has nonnegative Kodaira dimension, then it satisfies Vojta's Main Conjecture for general cycles (Conjecture 2.5).*
2. *If V has negative Kodaira dimension, then it is uniruled.*

Let v be a place of k, and choose an ample multiplicative height H on $X \times X$.

Then for any $\epsilon > 0$, there is a nonempty Zariski open subset $U(\epsilon)$ of X and a positive real constant C such that

$$dist_v(P, Q) > CH(P, Q)^{-\epsilon}$$

for all $P, Q \in U(k)$.

Proof: Let v be a place of k. Let Y be the variety $X \times X$, and let D be the diagonal on Y. Let h be an ample logarithmic height on X, and define an ample logarithmic height on Y by $h(P, Q) = h(P) + h(Q)$. By Vojta's Main Conjecture for general cycles (Conjecture 2.5), there is a proper Zariski closed subset $Z \subset Y$ such that

$$m_{D,v}(P, Q) \leq \epsilon h(P, Q) + O(1) \tag{3}$$

for every $(P, Q) \in Y(k) - Z(k)$. In what follows, we assume that Z is chosen to be the minimal closed subset such that equation (3) is satisfied.

Let V be an irreducible component of Z. We will show that V does not surject onto X via both projections from Y unless $V = D$. Thus, assume that V does surject onto X via both projections. Then $\dim V \geq \dim X$. Let $E = V \cap D$. If $E = V$, then $V = D$, so we may assume that E is a cycle with positive codimension on V. By, for example, Theorems 3.26 and 3.27 from [6], there is a smooth variety $\tilde{V}$ and a dominant birational map $\pi \colon \tilde{V} \to V$ such that π is an isomorphism away from $N = E \cup \operatorname{Sing}(V)$, and such that the induced reduced cycle M of $\pi^* N$ is contained in a divisor with normal crossings.

If $\tilde{V}$ has negative Kodaira dimension, then by assumption it is uniruled. Since it is birational to a subvariety of $X \times X$ that surjects onto

X via both projections, this implies that X is also uniruled. Since X has nonnegative Kodaira dimension, this is impossible.

Thus, we may assume that $\tilde{V}$ has nonnegative Kodaira dimension. This means that some multiple nK of the canonical divisor of $\tilde{V}$ must admit a global section, and so the height h_K is bounded below by a constant away from the base locus of nK. In addition, if M is the sum of the irreducible components of π^*N (i.e., M is the reduced divisor induced by π^*N), then outside a proper Zariski closed subset, we have:

$$m_S(\pi^*N, P) + h_K(P) \leq \alpha(m_S(M, P) + h_K(P)) + O(1)$$

for some positive integer α. Thus, if Vojta's Main Conjecture (Conjecture 2.5) is true for M, it must also be true for π^*N. (Note that M is, as noted above, contained in a normal crossings divisor.)

Thus, by Vojta's Main Conjecture applied to π^*N, we find a proper Zariski closed subset $\tilde{Z}$ of $\tilde{V}$ such that for all $Q \in \tilde{V}(k) - \tilde{Z}(k)$, we have:

$$m_{\pi^*N,v}(Q) \leq \epsilon h(Q) + O(1)$$

where $h(Q)$ in this case denotes $h(\pi(Q))$, since this yields a big height on $\tilde{V}$. Since E is an effective cycle whose support is contained in the support of the effective cycle N, we see that

$$m_{E,v}(P) \leq m_{N,v}(P) \leq m_{\pi^*N,v}(\pi^{-1}(P))$$

for all $P \in V(k) - N(k)$. (Recall that π is an isomorphism away from N.) It therefore follows that:

$$m_{E,v}(P) \leq \epsilon h(P) + O(1)$$

for all $P \in V(k) - N(k)$. Noting that $m_{D,v}(\pi(Q)) = m_{E,v}(Q) + O(1)$ for $Q \in \tilde{V}(k) - E(k)$ gives

$$m_{D,v}(P, Q) \leq \epsilon h(P, Q) + O(1)$$

for all k-rational points (P, Q) in some nonempty open subset of V. But this is precisely equation (3), which contradicts the minimality of V. We conclude that V does not surject onto X via both projections unless $V = D$.

Let W be the union of all proper closed subsets H of X such that H is the projection of some irreducible component of Z. (Recall that Z is the exceptional subset of X derived from Vojta's Main Conjecture.) Let $U' = X - W$. Then the intersection of $U' \times U'$ with the exceptional set $Z \subset X \times X$ is a subset of the diagonal, so for each $P, Q \in U'(k)$ with $P \neq Q$, we have $m_{D,v}(P, Q) < \epsilon h(P, Q) + c$. Taking the reciprocal exponential of both sides of that equation yields:

$$\text{dist}_v(P, Q) > CH(P, Q)^{-\epsilon}$$

as desired. ♣

Remark 3.2. If we make the stronger conjecture that the exceptional set Z in Vojta's Main Conjecture can be chosen independently of the field k (provided that k contains a certain fixed number field), then so too can the open set $U(\epsilon)$ in the Repulsion Principle.

Note that the Repulsion Principle is true unconditionally for subvarieties of abelian varieties. This is because Vojta's Main Conjecture for subvarieties of abelian varieties was proven by Faltings ([3], Theorem 2). Indeed, Faltings proves the Vojta inequality on a cofinite subset of an abelian variety, rather than merely a nonempty open subset. It is then immediately clear that the exceptional subsets of $X \times X$ do not surject onto X, and so the Repulsion Principle follows easily.

4. Vojta's Conjecture implies the Batyrev-Manin Conjecture for $K3$ surfaces

Before we can prove the implication in the title of this section, we must describe the relevant conjecture of Batyrev and Manin. For context and motivation of this conjecture, we refer the reader to [1].

Let X be a smooth, projective, algebraic variety defined over a number field k, and let L be an ample divisor on X. Choose a (multiplicative) height function H_L on V corresponding to L, and let W be any subset of X. We define the counting function for W by:

$$N_{W,L}(B) = \#\{P \in W \cap X(k) \mid H_L(P) < B\}$$

for any positive real number B. Batyrev and Manin [1] have made a series of conjectures about the behaviour of $N_{W,L}$. In the case that X is a $K3$ surface, their conjecture is as follows:

Conjecture 4.1 (Batyrev-Manin Conjecture for $K3$ Surfaces)**.** *Let $\epsilon > 0$ be any real number and L any ample divisor on X. Then there is a non-empty Zariski open subset $U(\epsilon) \subset X$ such that*

$$N_{U(\epsilon),L}(B) = O(B^{\epsilon}).$$

In fact, this principle should apply much more broadly than just to $K3$ surfaces. We have the following general result:

Theorem 4.2. *Let X be a smooth, projective variety of nonnegative Kodaira dimension, defined over a number field k. For any smooth, projective variety V birational to a subvariety of $X \times X$, make the following two assumptions:*

(1) *If V has nonnegative Kodaira dimension, assume that Vojta's Main Conjecture for general cycles (Conjecture 2.5) is true for V.*

(2) *If V has negative Kodaira dimension, assume that V is uniruled.*

Then for every $\epsilon > 0$ and ample divisor L on X, there is a nonempty Zariski open subset $U(\epsilon) \subset X$ such that

$$N_{U(\epsilon),L}(B) = O(B^{\epsilon})$$

Proof: Let v be an archimedean place of k, and let n be the real dimension of the variety X_v over the associated completion k_v of k. (That is, $n = \dim X$ if v is a real place, and $n = 2\dim X$ if v is a complex place.) By the Repulsion Principle, there is a nonempty Zariski open subset U of X and a positive real constant C such that for all P and Q in $U(k)$, we have:

$$\mathrm{dist}_v(P,Q) > CH(P,Q)^{-\epsilon/2n}$$

for all sufficiently large real numbers B, where $H(P,Q) = H(P)H(Q)$ for some fixed ample multiplicative height H on X.

Let A be the set $A = \{P \in U(k) \mid H(P) \leq B\}$. Then we have

$$\mathrm{dist}_v(P,Q) > CB^{-\epsilon/n}$$

for any pair of points P and Q in A. Since A lies in a real manifold of finite dimension n and finite volume, it follows that:

$$\#A = O(B^{\epsilon})$$

as desired. ♣

Remark 4.3. If we make the stronger conjecture that the exceptional set Z can be chosen to be the same for any number field k, then the exceptional sets $U(\epsilon)$ in Theorem 4.2 (and in Corollary 4.4 below) can also be chosen to be the same. Note, however, that they will still depend on ϵ, and that as $\epsilon \to 0$, larger and larger number fields k may be required to make the uniform choice of $Z(\epsilon)$ minimal.

In fact, if X is a surface, then it is easy to see that the exceptional subset $Z(\epsilon)$ must be the union of all rational curves of degree at most $2/\epsilon$. This follows immediately from the fact that the curves with at least B^{ϵ} points of height at most B are precisely the rational curves defined over k, with at least one rational point, and with degree at most $2/\epsilon$.

It is a deep theorem of Miyaoka ([9]) that any smooth, projective threefold with negative Kodaira dimension is uniruled. Since curves and surfaces with negative Kodaira dimension are also well known to be uniruled, we deduce the following corollary of Theorem 4.2:

Corollary 4.4. *Let X be a smooth, projective surface of nonnegative Kodaira dimension, defined over a number field k. Assume that Vojta's Main Conjecture for general cycles (Conjecture 2.5) is true for any variety birational to a subvariety of $X \times X$. Then for every $\epsilon > 0$ and ample divisor L on X, there is a nonempty Zariski open subset $U(\epsilon) \subset X$ such that*

$$N_{U(\epsilon),L}(B) = O(B^{\epsilon}).$$

In particular, Vojta's Main Conjecture implies the Batyrev-Manin Conjecture for $K3$ surfaces.

5. Manin's Rational Curve Conjecture

One can almost generalize the arguments here to prove the Rational Curve Conjecture of Manin ([7]), if one further assumes a conjecture implied by the Minimal Model Program. Here is Manin's conjecture:

Conjecture 5.1 (Rational Curve Conjecture)**.** *Let U be a nonempty Zariski open subset of a smooth, projective, algebraic variety X defined over a number field k, and let L be an ample divisor on X. If there are positive real constants δ and c such that $N_{U,L}(B) \geq cB^{\delta}$ for infinitely many arbitrarily large positive real numbers B, then U contains a nonempty open subset of a rational curve C, defined over k and containing a dense set of k-rational points.*

We will not quite be able to prove this. Instead, we will prove (modulo two conjectures) the slightly weaker assertion that U must contain a nonempty open subset of a rational curve defined over some finite extension of k.

Theorem 5.2. *Let U be a nonempty Zariski open subset of a smooth, projective algebraic variety X defined over a number field k, and let L be an ample divisor on X. For any smooth, projective variety V birational to a subvariety of $X \times X$, make the following two assumptions:*

1. *If V has nonnegative Kodaira dimension, assume that Vojta's Main Conjecture for general cycles (Conjecture 2.5) is true for V.*
2. *If V has negative Kodaira dimension, assume that V is uniruled.*

If there are positive real constants δ and c such that $N_{U,L}(B) \geq cB^{\delta}$ for infinitely many arbitrarily large positive real numbers B, then U contains a nonempty open subset of a rational curve C (not necessarily defined over k).

Proof: First, note that if the Kodaira dimension of X is negative, then the result is trivially true. Thus, in what follows, we assume that X has nonnegative Kodaira dimension. In particular, we may assume that some multiple nK of the canonical class admits a global section, and therefore that any height h_K associated to the canonical class is bounded from below by a constant away from the base locus of nK.

We will induce on the dimension of X. Theorem 5.2 is clearly true if $\dim X = 1$, so assume that it is true for all varieties of dimension less than X. By Theorem 4.2, there is a nonempty Zariski open subset $U \subset X$ such that

$$N_{U,L}(B) = O(B^\epsilon).$$

By comparing counting functions, this implies for small enough ϵ that there is some nonempty closed subset Y of X which is not contained in U. Moreover, we must have $\#\{P \in Y(k) \mid H(P) \leq B\} > CB^\delta$ for infinitely many arbitrarily large positive real numbers B and some real positive constant C. By the induction hypothesis, $Y \subset U$ must contain a nonempty open subset of a rational curve, as desired. ♣

As noted previously ([9]), the assumption that every variety of negative Kodaira dimension is uniruled is known to be true in dimension at most three, so the Rational Curve Conjecture for surfaces follows simply from Vojta's Main Conjecture. In higher dimensions, the standard conjectures of the Minimal Model Program imply that every variety of negative Kodaira dimension is uniruled. The Hard Dichotomy Theorem implies that the result of the Minimal Model Program applied to any variety of negative Kodaira dimension is a Mori fibre space. It is a theorem of Miyaoka and Mori ([10]) that any Mori fibre space is uniruled. For details, see for example [8], section 3.2.

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