Vojta's Main Conjecture for blowup surfaces

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Abstract

In this paper, we prove Vojta's Main Conjecture for split blowups of products of certain elliptic curves with themselves. We then deduce from the conjecture bounds on the average number of rational points lying on curves on these surfaces, and expound upon this connection for abelian surfaces and rational surfaces.

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1 Introduction

Vojta has formulated many deep conjectures about the arithmetic of algebraic varieties [Vo]. In this paper, we will be primarily concerned with the following conjecture [Vo, Conjecture 3.4.3].

Conjecture 1.1 (Vojta's Main Conjecture) *Let X be a smooth algebraic variety defined over a number field k, with canonical divisor K. Let S be a finite set of places of k. Let L be a big divisor on X, and let D be a normal crossings divisor on X. Choose height functions h_K and h_L for K and L, respectively, and define a proximity function $m_S(D,P)=\sum_{v\in S} h_{D,v}(P)$ for D with respect to S, where $h_{D,v}$ is a local height function for D at v. Choose any $\epsilon>0$. Then there exists a nonempty Zariski open set $U=U(\epsilon)\subset X$ such that for every k-rational point $P\in U(k)$, we have the following inequality:*

$$m_S(D,P)+h_K(P)\leq \epsilon h_L(P) \tag{1}$$

We recall the definition of big:

Definition 1.2 *Let L be a divisor on a smooth algebraic variety V. The dimension of L is the integer $\ell = \dim L$ satisfying $\ell(nL) \gg\ll n^\ell$ for n sufficiently divisible. If $\mathcal{L}(nL)$ is always empty, define $\dim L = -\infty$.*

A divisor L is big if and only if $\dim L = \dim V$.

Big divisors are characterised by the following proposition of Kodaira ([KO, Appendix]):

Proposition 1.3 *Let L be a divisor on a smooth projective variety V. Then L is big if and only if nL can be written as a sum of an ample divisor and an effective divisor for some sufficiently large integer n.*

If X is a curve of genus $g > 1$, or indeed any variety of general type, then the canonical divisor K itself is big, so choosing $\epsilon < 1$, $D = 0$, and $L = K$ in Conjecture 1.1 gives a dense open subset U on which $h_K(P) \leq 0$ for all $P \in U(k)$. This immediately implies that $U(k)$ is finite, so that $X(k)$ must be entirely contained in some Zariski closed proper subset. Thus, if X is a curve, $X(k)$ itself must be finite, which is precisely the statement of Faltings' Theorem.

However, very little has been proven about Vojta's conjectures in general, so the above argument is not likely to produce a new unconditional proof of Faltings' Theorem. It is more likely that Vojta's conjectures will be proven for other varieties first, and indeed some progress in that direction has already been made by Roth, Schmidt, Schlickewei, and others for $\mathbf{P}^n$, and Faltings has indeed already proven Conjecture 1.1 for abelian varieties. (See [Vo] and [EE] for an overview of known results.)

Note that the full strength of Conjecture 1.1 is not necessary above, since the term $m_S(D, P)$ is neglected by setting $D = 0$. In general, setting $D = 0$ yields the following (possibly) weaker conjecture:

Conjecture 1.4 *Define X, K, k, L, h_L, and h_K as in Conjecture 1.1, and choose any $\epsilon > 0$. Then there exists a nonempty Zariski open set $U = U(\epsilon) \subset X$ such that for every k-rational point $P \in U(k)$, the following inequality holds:*

$$h_K(P) \leq \epsilon h_L(P) \tag{2}$$

In §2, we prove the special case of Conjecture 1.4 where X is a blowup of a product $C \times C$, where C is an elliptic curve whose rational points $C(k)$ form a rank one module over the endomorphism ring $\mathrm{End}_k(C)$. From this and a result of Faltings [Fa], we deduce the corresponding special case of Conjecture 1.1. We further deduce that many curves on X have only finitely many rational points (Theorem 2.4) and that for certain pencils $\mathcal{P}$ of curves on X, the average number of rational points on curves in $\mathcal{P}$ is finite and can be effectively computed.

In §3, we describe how specialising Conjecture 1.4 to rational surfaces implies Faltings-type theorems for curves of general type (Theorems 3.4 and 3.5). Furthermore, we can get (conjectural) quantitative bounds on the average number of points on curves as they vary in plane pencils (Theorem 3.6).

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2 Abelian Surfaces

Let C be an elliptic curve defined over a number field k, and let $A = C \times C$. Let X be a geometrically smooth, projective k-scheme and $f\colon X \to A$ a birational k-morphism. Assume further that the exceptional set E of f satisfies $f(E) \subset Z$ for some finite set Z of k-rational points of A. Let L be a big divisor on X, and denote by K the canonical divisor on X. We may then choose height functions h_K and h_L associated to K and L, respectively.

In this section, we will assume either that $C(k)$ has rank one, or that $C(k)$ has rank two and k-rational complex multiplication. In this case, if $X = A$, Conjecture 1.4 is trivially true, since $K = 0$ and $h_L(P)$ is uniformly bounded below. However, if $X \neq A$, then $K \neq 0$, and the conjecture becomes non-trivial. The main result of this section is the following.

Theorem 2.1 (Vojta's Main Conjecture with $D = 0$) *Let X, L, and K be as above, and choose height functions h_L and h_K. For any $\epsilon > 0$, there is an effectively computable Zariski closed proper subset $V(\epsilon)$ such that for all k-rational points $P \in (X - V(\epsilon))(k)$, we have the following inequality:*

$$h_K(P) \leq \epsilon h_L(P) + O(1)$$

where the implied constant in the $O(1)$ is independent of P.

Proof: First, we will make several reductions. Let $Z = \{z_1, \ldots, z_n\}$ be the set of points of A over which f is not an isomorphism. Then there is an effective divisor F on X and positive integers $a_1, \ldots, a_n$ such that $K + F = a_1 f^{-1}(z_1) + \ldots + a_n f^{-1}(z_n)$. (To see this, take a_i to be the maximum multiplicity of any component of the fibre $f^{-1}(z_i)$ in K, and note that K is supported entirely on the exceptional set of f.) Conjecture 1.4 therefore becomes:

$$(\sum_i a_i h_{f^{-1}(z_i)}(P)) - h_F(P) \leq \epsilon h_L(P) + O(1) \tag{3}$$

Since F is effective, it suffices to assume that $F = 0$. It clearly also suffices to prove the theorem for $n = 1$ and $a_1 = 1$, since if X_1 and X_2 are two blowups of A at a single point, then X_1 is isomorphic to X_2. Therefore, by adjusting ϵ, we can reduce to the case $n = 1$ and $a_1 = 1$, in which case f is the blowup of A at a single point, say, the origin. Denote the exceptional divisor of f by E.

Furthermore, it suffices to prove the theorem for a single big divisor L, since if L' is another big divisor on X, then there are constants c_1 and c_2 such that up to $O(1)$, $c_1 h_L(P) \leq h_{L'}(P) \leq c_2 h_L(P)$ for all $P \in U(k)$ for some nonempty Zariski open set $U \subset X$. Thus, without loss of generality, we may assume that $L = f^*(\pi_1^*(O) + \pi_2^*(O))$, where $\pi_i : C \times C \to C$ is the ith projection, and O is the identity element of C. This choice of L is big but not ample.

Therefore, we have reduced Conjecture 1.4 to the following inequality:

$$h_E(P) \leq \epsilon h_L(P) + O(1)$$

Let v be any place of k. Then the local height $h_{E,v}$ can be written away from E as $\min\{h_{(f\circ\pi_1)^*Y,v}, h_{(f\circ\pi_2)^*Y,v}\}$ [Vo, Lemma 2.5.2], where $h_{Y,v}$ is some choice of local height function for $Y = (O)$, the divisor class of the identity element of C. Since the conclusion of the theorem is independent of the choice of height function, it suffices to assume that the exceptional height is given locally by this formula. Similarly, it suffices to assume that the height function h_Y is the canonical height function. For ease of notation, we will denote h_Y and $h_{Y,v}$ by h and h_v, respectively. Finally, we may assume that C is given in Weierstrass form $y^2 = x^3 + ax + b$ for some integers a, b.

Fix a k-rational point Q_0 on C of infinite order, and let e be the least integer such that for every element $P \in C(k)$, there exists an element $\phi \in$

$\mathrm{End}_k(C) = R$ such that $eP = \phi(Q_0)$. Choose $\epsilon > 0$, and choose a positive integer M large enough so that $\epsilon > 2/M$, and assume that $P = f^{-1}(P_1, P_2)$ satisfies $Mh_E(P) \geq h_L(P)$.

Lemma 2.2 *Let v be a finite place of k at which C and every element of $R = End(C)$ have good reduction. Then for any two points $P_1, P_2 \in C(k)$ and any two elements $m_1, m_2 \in R$, $h_v(m_1P_1 + m_2P_2) \geq h_{E,v}(f^{-1}(P_1, P_2))$, provided that $m_1P_1 + m_2P_2 \neq 0$.*

Proof: At a prime of good reduction, we have $2h_v(P) = \mathrm{ord}_v(x(P))$. If C does not have CM, the lemma then follows immediately from Theorem III.1.1 of [La], which essentially states that the set of points $P \in C(k)$ with $h_v(P) \geq x$ (together with the identity) forms a group for any real number $x > 0$.

If C does have CM, then one must show in addition that $h_v(\phi(P)) \geq h_v(P)$ for any endomorphism $\phi \in R$, provided that $\phi(P) \neq 0$. For this, note that at a good prime, we have

$$h_v(P) = \sum_j (\log N_{k/\mathbf{Q}} v) i(P_j, \overline{P} \cdot \overline{O}, \mathcal{C})$$

where $N_{k/\mathbf{Q}}$ denotes the usual norm, $i(W, X \cdot Y, Z)$ denotes the intersection multiplicity of a component W of the intersection of subschemes X and Y on a scheme Z (see for example [Fu, §7]). Let $\mathcal{C}$ denote the smooth Weierstrass model of C over $\mathcal{O}_v$ (the local ring of k at v), and $\overline{P}$ and $\overline{O}$ denote the closures of P and the origin O in $\mathcal{C}$. The P_j are the points of the intersection of $\overline{P}$ and $\overline{O}$.

Since $\phi \in R$, it induces an isomorphism $\phi_P \colon \overline{P} \to \overline{\phi(P)}$ over $\mathcal{O}_v$. From [Fu, Example 20.2.2], we may apply the projection formula [Fu, Example 7.1.9] to get for each component Q of the intersection of $\overline{\phi(P)}$ and $\overline{O}$:

$$i(Q, \overline{\phi(P)} \cdot \overline{O}, \mathcal{C}) = i(\phi_P^{-1}(Q), \overline{P} \cdot \overline{O}, \mathcal{C})$$

If $\phi^{-1}(Q) \notin \overline{O}$, then $i(\phi_P^{-1}(Q), \overline{P} \cdot \overline{O}, \mathcal{C}) = 0$. ¿From this we see that

$$\sum_Q i(Q, \overline{\phi(P)} \cdot \overline{O}, \mathcal{C}) \geq \sum_j i(P_j, \overline{P} \cdot \overline{O}, \mathcal{C})$$

since for each j, P_j satisfies $\phi(P_j) \in \overline{\phi(P)} \cap \overline{O}$. It immediately follows that $h_v(\phi(P)) \geq h_v(P)$. ♣

Let Γ be the submodule of $C(k)$ generated by P_1 and P_2, and let P_0 be the element of Γ which minimises $h(P_0)$. Let S denote the set of places of k which are infinite or at which C has bad reduction. From the lemma, it follows that $h_{E,v}(P) \leq h_v(P_0)$ for all finite places $v \notin S$. If for any divisor D on X we define $h_D^g = \sum_{v \notin S} h_{D,v}$, then we get $h_E^g(P) \leq h^g(P_0)[= h_Y^g(P_0)]$.

By Siegel's Theorem [Si, Theorem IX.3.1], for any place v, there are only finitely many points Q in $C(k)$ such that $(2M)h_v(Q) \geq h(Q)$. For any positive real number N, let $W(N)$ denote the finite set of points $f^{-1}(P_1, P_2) \in (X - E)(k)$ for which $N \sum_{v \in S} h_v(P_i) > h(P_i)$ for both $i = 1, 2$. Then if $P \notin W(2M) \cup E(k)$, we must have

$$\begin{aligned} h_E(P) &= \sum_{v \in S} h_{E,v}(P) + h_E^g(P) \\ &\leq \frac{1}{2M} h_L(P) + h^g(P_0) \\ &\leq \frac{1}{2M} h_L(P) + h(P_0) \end{aligned}$$

By assumption, we know that $Mh_E(P) \geq h_L(P)$, so

$$(1/2M)h_L(P) + h(P_0) \geq (1/M)h_L(P)$$

and hence $h(P_0) \geq (1/2M)h_L(P)$. For $i = 1, 2$, write $eP_i = m_i Q_0$ for some $m_i \in R$. Then there is an element $m_0 Q_0 \in e\Gamma$ such that:

$$\log N_R(m_0)/N_R(\gcd(m_1, m_2)) \leq \log B \tag{4}$$

where $B > 1$ is a real number depending only on the ring R and N_R denotes the norm map $N_R \colon R \to \mathbf{Z}$.

Thus, we have $(1/2M)h_L(P) \leq h(P_0) \leq (1/e)BN_R(m_0)h(Q_0)$. But:

$$\begin{aligned} (1/2M)h_L(P) &= (1/2Me)(h(m_1 Q_0) + h(m_2 Q_0)) \\ &= ((N_R(m_1) + N_R(m_2))/2Me)h(Q_0) \end{aligned}$$

so it follows that

$$\frac{N_R(m_1) + N_R(m_2)}{N_R(m_0)} \leq 2MB \tag{5}$$

Inequality (5) says precisely that the pair

$$(N_R(m_1)/N_R(m_0), N_R(m_2)/N_R(m_0))$$

is a rational point lying inside the triangle in the first quadrant of the xy-plane bounded by the line $x + y \leq 2MB$ and the coordinate axes. Inequality (4) implies that in lowest terms, the denominators are bounded above by B, so there are only finitely many choices for such pairs.

Since R is either $\mathbf{Z}$ or an order in an imaginary quadratic field, its unit group is finite. Therefore, the norm map N_R is finite-to-one. Again by inequality (4), there are therefore only finitely many choices for the pair $(m_1/m_0, m_2/m_0) \in R^2$. We may choose a finite set of lines $L_1, \ldots, L_n$ through the origin in R^2 so that each possible pair $(m_1/m_0, m_2/m_0)$ lies on one of the lines.

For any pair $\alpha = (a, b) \in R^2$, denote by D_α the image of the map $g_\alpha : C \to C \times C$ defined by $g_\alpha(X, Y) = (aX, bY)$. Then P must lie on one of the finitely many curves $f^{-1}(D_{\alpha_i})$, where $\alpha_i = (a_i, b_i)$ is any point on L_i. If we set $V(\epsilon)$ to be the union of those curves with the set $W(2M) \cup E(k)$, then we have proven the theorem.

One final note about effectivity. The curves $f^{-1}(D_{\alpha_i})$ and E are obviously effectively computable, but $W(2M)$ is a little more elusive, since it is a byproduct of Siegel's Theorem. There are effective versions of Siegel's Theorem however (see for example [Ba]). ♣

Remark: It ought to be a straightforward matter to extend Theorem 2.1 to the case in which C is an arbitrary abelian variety whose k-rational points are a rank one module over End(C). In the proof, Siegel's Theorem would have to be replaced by Faltings' result quoted below (Theorem 2.6), which would destroy the effectivity, although only for the zero-dimensional part of the exceptional set $V(\epsilon)$.

Now we will show how our main result can be applied to rational points on curves. Let A/k be any abelian surface, defined over a number field k. Let $\mathcal{P}$ be a k-rational pencil of curves on A. We have the following results:

Theorem 2.3 *Assume Conjecture* 1.4 *holds for all blowups of an abelian surface A. Let $\mathcal{P}$ be any pencil of curves on A. Then the average number of k-rational points on a general curve in $\mathcal{P}$ is finite, if the curves in $\mathcal{P}$ are ordered by height.*

Corollary 2.4 (Faltings' Theorem for curves on A) *Assume that Conjecture* 1.4 *holds for all blowups of an abelian surface A. Then for any pencil $\mathcal{P}$ on A, almost all curves of $\mathcal{P}$ have only finitely many k-rational points.*

Remark: Note that for any curve C on A, we can construct a pencil $\mathcal{P}$ of curves on A such that C is a component of some curve in $\mathcal{P}$.

Proofs: We will prove both results at once. Let $\pi: X \to A$ be a resolution of the rational map corresponding to $\mathcal{P}$ as in §2, and let $f: X \to \mathbf{P}^1$ be the corresponding fibration. Let C be the divisor class of $\mathcal{P}$, and let $L = \pi^*(C)$ be the corresponding big divisor on X. Let E be the exceptional locus of π. By hypothesis, we know Conjecture 1.4 for X, so we get:

$$h_K(P) \leq \epsilon h_L(P) \tag{6}$$

for some dense open set $U(\epsilon)$ and any $P \in U(\epsilon)$. Since $K - E$ is effective, it follows that $h_E(P) \leq \epsilon h_L(P)$. (The set $U(\epsilon)$ may need to shrink slightly for this.)

Let $F = f^*(\mathcal{O}(1))$ be the divisor class of a fibre of f. Then $F - L + mE$ is effective for some positive integer m, so by choosing $\epsilon < 1/m$ we get:

$$h_F(P) \geq \alpha h_L(P) \tag{7}$$

for some $\alpha > 0$. Since L is big, this implies that there is a dense open subset of V on which h_L satisfies a Northcott-type finiteness theorem. Thus, there is a dense open subset U of V such that for any real number B, the set $\{P \in U(k) \mid h_F(P) \leq B\}$ is finite. Since $h_F(P)$ depends only on the fibre of f on which P lies, it follows that every fibre of f not disjoint from U has only finitely many rational points.

Moreover, the set $\{P \in U(k) \mid \alpha h_L(P) \leq B\}$ has cardinality $\gg\ll (\log B)^{r/2}$, where r is the Mordell-Weil rank of A over k, and so the same will be true for $\{P \in U(k) \mid \alpha h_L(P) \leq B\}$. The set $\{P \in \mathbf{P}^1(k) \mid h_{\mathcal{O}(1)}(P) \leq B\}$ has cardinality $\gg\ll B^2$ [Sch]. Therefore, there must be a density one set of points $P \in \mathbf{P}^1(k)$ whose preimages under f contain no k-rational points in $U(k)$. Since only finitely many fibres of f are not disjoint from $U(k)$, the average number of points on curves in $\mathcal{P}$ must be finite. ♣

It would be nice to be able to apply Theorem 2.1 directly to Theorems 2.4 and 2.3 to deduce unconditional results about rational points on curves. Unfortunately, these theorems only apply when the basepoints of the pencil are k-rational, and if we extend k to include the basepoints of the pencil, the hypothesis on the rank of $C(k)$ may become false. However, there are some cases in which we do have a basepoint locus which splits completely over k. For instance, we have the following result:

Corollary 2.5 *Let C be an elliptic curve of rank one over a number field k, or of rank two over k with k-rational complex multiplication. Let $A = C \times C$, and let F_1 and F_2 be linearly equivalent effective divisors whose components are fibres of the two projection maps $A \to C$. Assume F_1 and F_2 intersect properly, and let $\mathcal{P}$ be the pencil of curves through F_1 and F_2. Then every component of a curve in $\mathcal{P}$ either has genus 1, or has finitely many k-rational points. Moreover, there exists an effectively computable Zariski dense open set U such that the average number of k-rational points in $Z(k) \cap U(k)$ is zero, where Z varies amongst all curves in $\mathcal{P}$, ordered by height.*

Remarks: Note that divisors described in the theorem are easy to construct. Let D_1 be any very ample divisor on C which is a sum of k-rational points. Then we can certainly find a disjoint divisor D_2 which is linearly equivalent to D_1 and is also a sum of k-rational points. We may then set $F_i = \pi_1^* D_i + \pi_2^* D_i$.

Note also that every curve in $\mathcal{P}$ has several rational points, corresponding to the basepoints of the pencil. These points are not in $U(k)$, which consists of the intersection of the set $U(1/2)$ from Theorem 2.1 with its translates by the basepoint locus of $\mathcal{P}$.

This result is related to the results of Manin and Demjanenko (see for example [Se, §5.2]). They prove, for example, that any curve of genus greater than one which admits more than r different k-rational morphisms to an elliptic curve of k-rank r has only finitely many rational points. The first part of Corollary 2.5 is contained in this result, but the second part is not.

Proof: By blowing up the basepoint locus of $\mathcal{P}$ to get a surface $p\colon X \to A$, we obtain a map $f\colon X \to \mathbf{P}^1$ whose fibres are the curves in $\mathcal{P}$. Let E be the exceptional locus of p, and let $F = f^*(\mathcal{O}(1))$. Then the canonical divisor on X is just E, since the basepoint locus of $\mathcal{P}$ is reduced, and $F + E$ is big. (Indeed, it's linearly equivalent to $p^* F_1$ and $p^* F_2$.) By Theorem 2.1, there exists an effectively computable dense open subset U of X such that for every $P \in U(k)$, we have the following inequality:

$$h_E(P) \leq (1/2) h_{F+E}(P)$$

This implies imediately that:

$$h_F(P) \geq (1/2) h_{F+E}(P)$$

Since $F + E$ is big, we may shrink U slightly so that the set $\{P \in U(k) \mid h_{F+E}(P) \leq B\}$ is finite for any real number B. Thus, the set $\{P \in U(k) \mid$

$h_F(P) \leq B\}$ must also be finite. But $h_F(P)$ depends only on $f(P)$, so it follows that every curve in $\mathcal{P}$ not disjoint from U must contain only finitely many k-rational points. Since the complement of U is a union of points and curves of genus 1, the first part of the corollary follows.

For the second part, we will compare the counting functions for U relative to h_F and h_{F+E}. We have:

$$\#\{P \in \mathbf{P}^1(k) \mid h_{\mathcal{O}(1)}(P) \leq B\} \gg\ll B^2$$
$$\#\{P \in U(k) \mid h_{F+E}(P) \leq B\} \gg\ll \log B \ll B^2$$

so since $h_F(P) = h_{\mathcal{O}(1)}(f(P))$, it follows that almost all fibres of f cannot have any k-rational points at all in U, and hence the average number of such rational points must be 0. ♣

Theorem 2.1 also implies the full strength of Conjecture 1.1 for X by way of a result of Faltings [Fa, Theorem 2]:

Theorem 2.6 (Faltings, 1991) *Let A be an abelian variety, h_L a height associated to an ample divisor L on A, and $\epsilon \geq 0$ any positive real number. Let D be any divisor on A. Then for all but finitely many k-rational points P on $A - D$, the following inequality holds:*

$$m_S(D, P) \leq \epsilon h_L(P)$$

Remark: Faltings proves this result only in the case that S contains a single place and D is irreducible, but this is obviously equivalent to the more general result. He also proves considerably more than this, but the above version will suffice for our purposes.

¿From this we can deduce the following corollary of Theorem 2.1:

Corollary 2.7 (Vojta's Main Conjecture) *Let X, L, and K be as in Theorem 2.1, and choose height functions h_L and h_K. Let D be any divisor on X, and fix a finite set S of places of k. For any $\epsilon > 0$, there is a Zariski closed set $V(\epsilon)$ such that for all k-rational points $P \in (X - V(\epsilon))(k)$, we have the following inequality:*

$$m_S(D, P) + h_K(P) \leq \epsilon h_L(P) + O(1)$$

where the implied constant in the $O(1)$ is independent of P.

Proof: First, find a divisor D' on $A = C \times C$ such that $f^*D' - D$ is effective. Then away from the exceptional set of f and the support of f^*D', we have $m_S(D', f(P)) = m_S(f^*D', P) \geq m_S(D, P) + O(1)$, so without loss of generality we may assume that $D = f^*D'$.

Next, recall that the conclusion of the theorem is independent from the choice of L, so we may choose L such that for some ample divisor L' on A, the divisor $L - f^*L'$ is effective. Thus, we may assume that $L = f^*L'$. Note that the L used in the proof of Theorem 2.1 is of this type.

This means that $h_L(P) = h_{L'}(f(P))$. By Theorem 2.6 applied to $f(P)$, it follows that there is only a Zariski closed set of points $f(P)$ satisfying $m_S(D', f(P)) > \epsilon h_{L'}(f(P))$, and therefore only a Zariski closed set of points P satisfying $m_S(D, P) > \epsilon h_L(P)$.

But Theorem 2.1 implies that there is only a Zariski closed set of points P satisfying $h_K(P) > \epsilon h_L(P)$. By adjusting ϵ and putting these two facts together, we deduce the Corollary. ♣

3 Rational Surfaces

Similar results linking Vojta's Main Conjecture to rational points on curves can be made for rational surfaces as well. The situation is more complicated, however.

Consider a pencil $\mathcal{P}$ of plane curves (not necessarily smooth) of degree $d \geq 4$ and geometric genus $g \geq 2$, defined over a number field k. This pencil defines a rational map $\mathbf{P}^2 \to \mathbf{P}^1$, which we may resolve into a smooth rational surface $f\colon X \to \mathbf{P}^2$ which admits a fibration $\pi\colon X \to \mathbf{P}^1$ whose fibres are the curves of $\mathcal{P}$.

Let L be the big class $f^*(H)$, where H is the class of a line in $\mathbf{P}^2$, and let $F = \pi^*\mathcal{O}(1)$ be the class of a fibre of π. Note that we may identify F as the strict transform in X of a general curve C of the pencil $\mathcal{P}$. Denote by K_X the canonical divisor of X. We have the following proposition:

Proposition 3.1 *Let C be a general curve of $\mathcal{P}$, and let m be the maximum multiplicity of C at any basepoint P_i of $\mathcal{P}$. Then $F + mK_X \geq (d - 3m)L$, with equality if $m = 1$.*

Proof: We have a birational morphism $f\colon X \to \mathbf{P}^2$, which can be decomposed into a sequence $X = X_0 \overset{f_0}{\to} X_1 \overset{f_1}{\to} \cdots \overset{f_{n-2}}{\to} X_{n-1} \overset{f_{n-1}}{\to} X_n = \mathbf{P}^2$ of blowups of points [GH, p. 510]. Define $\psi_i\colon X_i \to \mathbf{P}^2$ to be the composition of the maps $f_i, \ldots, f_{n-1}$. We proceed by backwards induction on i.

To start the induction, note that the canonical divisor of $\mathbf{P}^2$ is $-3H$, and a curve in $\mathcal{P}$ is dH, so the claim is trivially true. The inductive step is the following lemma:

Lemma 3.2 *Let $f_i\colon X_i \to X_{i+1}$ be as above, and let $K_i = K_{X_i}$ be the canonical divisor of X_i. Define F_i to be the strict transform of a general curve $C \in \mathcal{P}$ in X_i. If $F_{i+1} + mK_{i+1} \geq (d-3m)\psi_{i+1}^* H$, then $F_i + mK_i \geq (d-3m)\psi_i^* H$. If $m = 1$, then in addition, equality for $i+1$ implies equality for i.*

Proof of lemma: Let E_i be the exceptional divisor of f_i, and assume that $F_{i+1} + mK_{i+1} \geq (d-3m)\psi_{i+1}^* H$. We have $F_i = f_i^* F_{i+1} - rE_i$, where r is the multiplicity of the point $f_i(E_i)$ on F_{i+1}. In particular, since blowing up a point can never increase the multiplicity of a point on a curve, it follows that $F_i \geq f_i^* F_{i+1} - mE_i$. Note also that if $m = 1$, then $r = 1$, so in that case we may conclude by induction that $F_i = f_i^* F_{i+1} - E_i$.

On the other hand, we have $K_i = f_i^* K_{i+1} + E_i$ by [H, Ex. II.8.5]. Therefore, $mK_i = mf_i^* K_{i+1} + mE_i$. Putting all this together, we get:

$$\begin{aligned} F_i + mK_i &\geq (f_i^* F_{i+1} - mE_i) + (mf_i^* K_{i+1} + mE_i) \\ &\geq (d-3m) f_i^* \psi_{i+1}^* H \\ &= (d-3m)\psi_i^* H \end{aligned}$$

where each inequality becomes an equality if $m = 1$. The lemma follows. ♣

Since $F = F_0$, Proposition 3.1 follows from Lemma 3.2. ♣

Define the ramification divisor $E = K_X + 3L$ of f; this is an effective divisor supported on the exceptional locus of f. By setting $D = 0$ in Conjecture 1.1 and specialising to our current situation, we get that for all k-rational points P on some dense open set $U(\epsilon) \subset X$, we have:

$$h_E(P) \leq (3+\epsilon) h_L(P) + O(1) \tag{8}$$

Now define $E' = dL - F$. If a general curve of $\mathcal{P}$ is smooth, then we may take $m = 1$ in the previous lemma, and hence $E = E'$. In that case, the effectivity of F trivially but unconditionally implies the following weakening of inequality (8):

$$h_E(P) \leq d h_L(P) + O(1) \tag{9}$$

For any point $P \in X$, the height $h_F(P)$ is equal to $h_{\mathcal{O}(1)}(\pi(P))$. Therefore, for any bound $B > 0$, the set of points $P \in X(k)$ with $h_F(P) < B$ is precisely the set of points which lie on k-rational fibres of π lying over points of $\mathcal{O}(1)$-height at most B. Since $\mathcal{O}(1)$ is ample on $\mathbf{P}^1$, Northcott's theorem [Vo, Proposition 1.2.9.(g)] implies that there are only finitely many points $Q \in \mathbf{P}^1(k)$ with $h_{\mathcal{O}(1)}(P) < B$. Since all but finitely many k-rational fibres of π contain only finitely many k-rational points by Faltings' Theorem, we have the following theorem:

Theorem 3.3 *Let $U \subset X$ be the complement of the finitely many fibres of π whose geometric genus is at most 1. Then the set $\{P \in U(k) \mid h_F(P) < B\}$ is finite for every $B > 0$.*

If the fibres of π have a sufficiently large genus, then for a generic choice of pencil $\mathcal{P}$, every fibre of π will have only finitely many rational points. Thus, Theorem 3.3 gives an example of a non-big divisor F on a rational surface which nevertheless satisfies a Northcott-type theorem.

Conversely, weakened forms of inequality (8) very nearly imply Faltings' Theorem. For example, let m be the maximum multiplicity of a general curve of $\mathcal{P}$ at a basepoint, and let $3 < n < d/m$. (The inequality $n > 3$ is necessary to allow (8) to imply (10) below.) Then we have the following theorem:

Theorem 3.4 (Faltings' Theorem for general curves) *Assume that for any k-rational point P in some dense open subset $U \subset X$, we have the following inequality:*

$$h_E(P) \leq n h_L(P) + O(1) \tag{10}$$

Then for any pencil $\mathcal{P}$ of curves of degree $d > n$ in $\mathbf{P}^2$, almost all curves of $\mathcal{P}$ have only finitely many k-rational points. In particular, if we assume conjecture (8), then we may take $n = 3 + \epsilon$.

Proof: Assume without loss of generality that $U \cap E = \emptyset$. This would imply that for all $P \in U(k)$, $h_{dL-mE}(P) \geq (d-mn)h_L(P) + O(1)$. Since $mE - E'$ is effective, it then follows that $h_F(P) \geq (d-mn)h_L(P) + O(1)$. But $(d-mn)L$ is the pullback of the ample divisor $(d-mn)H$ on $\mathbf{P}^2$, so Northcott's theorem implies that for any bound $B > 0$, there are only finitely many $P \in U(k)$ with $h_{(d-mn)L}(P) < B$, and therefore only finitely many $P \in U(k)$ with

$h_F(P) < B$. This in turn implies that every fibre of π not disjoint from U can contain at most finitely many k-rational points. ♣

Remark: Note that we cannot take $n \leq 3$ in (10), since Theorem 3.4 applied to a pencil of smooth plane cubics would then imply that elliptic curves have only finitely many rational points. Since this is not generally true, the coefficient $3 + \epsilon$ in (8) is sharp.

To imply the full Faltings' Theorem, it remains only to show that for any curve C of genus at least two, there exists some rational fibration $\pi: X \to \mathbf{P}^1$ with corresponding open set U as above such that some fibre T of π is birational to C, and such that $T \cap U$ is nonempty. One way to assure this is to construct a pencil of curves whose generic member is $\overline{k}$-birational to C. If the pencil is defined over k, this will ensure that infinitely many curves in it are k-birational to C, and hence that at least one fibre of the corresponding map π will intersect U.

Unfortunately, this cannot generally be done for an arbitrary curve C of general type. However, we do have the following theorem:

Theorem 3.5 *Let $f(x,z)$ and $g(x,z)$ be coprime homogeneous polynomials. Let $n \in \mathbf{Q}$, and let m and d be positive integers satisfying $d/m \geq n \geq 3$. Let C be a curve of general type, dominating over $\overline{k}$ the plane curve D given by $y^a f(x,z) = g(x,z)$, where $\deg f + a = \deg g = d$. Assume no singularity of D has multiplicity greater than m. Then conjecture* (10) *implies that C contains only finitely many k-rational points.*

Proof: If C dominates a curve with only finitely many rational points, then it can only contain finitely many rational points itself. Therefore, it suffices to assume that C itself is the plane curve $y^a f(x,z) = g(x,z)$. Consider the pencil of curves $Ay^a f(x,z) = Bg(x,z)$. Clearly, almost all curves in this pencil are birational to C, so construct the corresponding rational fibration $p: X \to \mathbf{P}^1$. By Theorem 3.4, all but finitely many fibres of p have only finitely many k-rational points. But infinitely many fibres of p are k-birational to C, so therefore C can have only finitely many k-rational points. ♣

This class includes for example the Fermat curves $y^n = x^n + z^n$. There are also non-planar curves in this class. Let $p(x,z)$ be a squarefree, homogeneous polynomial of degree 6 with $p(a,1) \neq 0$. The curve $y^8 = (x - az)^2 p(x,z)$ has a single double point, and its normalisation has genus 17, and is therefore not planar. Therefore, its normalisation has only finitely many k-rational points by Theorem 3.5.

Moreover, let C be a curve which satisfies the hypotheses of Theorem 3.5, and let $f: C \to C'$ be an unramified cover. Then by [Vo, Thm 1.4.11], there is a finite extension l/k such that the l-rational points of C map surjectively onto the k-rational points of C'. Therefore, since there are only finitely many l-rational points on C, there can by only finitely many k-rational points on C'.

If we assume a slightly stronger version of inequality (10), then we may prove the following strengthening of Theorem 3.5, which roughly speaking states that the average plane curve has no k-rational points:

Theorem 3.6 *Let m be a positive integer, and assume that inequality* (10) *holds for some $n \in \mathbf{Q}$ satisfying $d - mn > 3/2$. Let $\mathcal{P}$ be any pencil of plane curves of degree d whose singular basepoints have multiplicity at most m. Then the average number of k-rational points on a general curve in $\mathcal{P}$ is finite, if the curves in $\mathcal{P}$ are ordered by height. Moreover, the average number of such points which correspond to points in the set $U(k)$ is 0, independent of the field k.*

Proof: If $d - mn > 3/2$, then we have $h_F(P) \geq \alpha h_L(P) + O(1)$ on some dense open set $U \subset X - \mathrm{Supp}(E)$, where $\alpha = d - mn > 3/2$. By Schanuel's theorem [Sch], the number of points $P \in (X - \mathrm{Supp}(E))(k)$ with $H_L(P)^\alpha \leq B$ is $\gg\ll B^{3/\alpha} \ll B^{2-\epsilon}$ for $\epsilon > 0$. (Note that Schanuel's theorem is stated in terms of multiplicative heights, which we denote by H. Hence, for any divisor D, we have $h_D = \log H_D$.) Therefore, the number of points $P \in U(k)$ with $H_F(P) \leq B$ is also $\ll B^{2-\epsilon}$.

However, the set $\{P \in U(k) | H_F(P) \leq B\}$ is just the set of points of $U(k)$ which map to a k-rational point of $\mathbf{P}^1$ with $\mathcal{O}(1)$-height at most B. Again by Schanuel's theorem, the set of points $P \in \mathbf{P}^1(k)$ with $H_{\mathcal{O}(1)}(P) \leq B$ is $\gg\ll B^2$. Therefore, the average number of k-rational points on the fibres of $p|_U$, ordered by height, is at most $\lim_{B\to\infty} B^{2-\epsilon}/B^2 = 0$. Since the number of k-rational points of $(X - U) \cap p^{-1}(x)$ is bounded independently of the choice of $x \in \mathbf{P}^1(k)$ (apart from the finitely many fibres of p which have some component disjoint from U), it follows that the average number of points on a general fibre is finite. ♣

In particular, if we choose $\mathcal{P}$ to have d^2 distinct, simple basepoints, then a general curve in the pencil will be smooth. Therefore, in that case, we may take $m = 1$ and $n < d - 3/2$.

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