CODIMENSION TWO INTEGRAL POINTS ON SOME RATIONALLY CONNECTED THREEFOLDS ARE POTENTIALLY DENSE

DAVID MCKINNON AND MIKE ROTH

Abstract. Let V be a smooth, projective, rationally connected variety, defined over a number field k, and let $Z \subset V$ be a closed subset of codimension at least two. In this paper, for certain choices of V, we prove that the set of Z-integral points is potentially Zariski dense, in the sense that there is a finite extension K of k such that the set of points $P \in V(K)$ that are Z-integral is Zariski dense in V. This gives a positive answer to a question of Hassett and Tschinkel from 2001.

1. Introduction

In [HT], as Problem 2.13 ("The Arithmetic Puncturing Problem"), Hassett and Tschinkel ask the following question:

Question 1.1. Let X be a projective variety with canonical singularities and Z an algebraic subset of codimension at least 2, all defined over a number field k. Assume that rational points on X are potentially dense. Are integral points on (X, Z) potentially dense?

Of course, the hypothesis that Z has codimension at least two cannot be removed, as there are countless well known examples of varieties with a dense set of rational points but a degenerate set of integral points if Z is a divisor. Note that when we say that an algebraic set has codimension at least two, we mean that every irreducible component has codimension at least two.

In [HT], they provide positive answers to this question in various cases, including toric varieties and products of elliptic curves. The purpose of this paper is to give a positive answer to this question for a large number of examples in dimension up to three.

The case for curves seems vacuous, but if one views a curve defined over a number field as an arithmetic surface, then one can choose Z to be an arithmetic zero-cycle, in which case there is something to prove. This is Lemma 2.3, and is a crucial technical tool for the paper.

For surfaces, the situation is more complicated, as it is unknown which surfaces have a potentially dense set of rational points. If the Kodaira dimension is negative, however – which is believed to be the case in which rational points are most plentiful – we give a positive answer to Question 1.1 in Theorem 3.1.

In the lengthiest and most difficult part of the paper, X will be a smooth, projective, rationally connected threefold. It is a theorem of Mori ([Mo]) that there is a birational map $f\colon X \dashrightarrow V$, where V is a normal projective threefold with only $\mathbb{Q}$-factorial and terminal singularities with a morphism $\pi\colon V \to Y$ of one of the following three types:

(a) The variety Y is a normal projective surface with at most rational singularities, and π makes V a conic bundle over Y.
(b) The variety Y is isomorphic to $\mathbb{P}^1$, and a general fibre of the morphism Y is a smooth del Pezzo surface.
(c) The variety Y is a point, and $\mathrm{Pic}(V) \cong \mathbb{Z}$.

This list provides a natural set of examples on which to test Question 1.1. Indeed, in light of Lemma 2.2, a positive answer to Question 1.1 for the varieties listed above will provide a positive answer for any blowup of such varieties, which constitutes a huge proportion of all smooth, rationally connected threefolds. In this paper, we will deal with examples from cases b and c. Specifically, we prove the following theorems:

Theorem 1.2. *4.1 Let X be a complex Fano threefold of Picard rank one and index at least two. Assume that X is defined over a number field k, and let Z be an algebraic subset of X of codimension at least two. If X is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$, then we make the further assumption that Z does not contain the unique basepoint P of the square root of the anticanonical linear system. Then the Z-integral points of X are potentially Zariski dense.*

Theorem 1.3. *6.1 Let k be a number field. Let X be a smooth threefold with a map $\pi\colon X \to \mathbb{P}^1$ whose generic fibre is a del Pezzo surface of degree at least three, all defined over k. Let $Z \subset X$ be an algebraic subset of codimension at least two. Then the Z-integral points of X are potentially Zariski dense.*

The rest of the paper is structured as follows. In section 2, we make some preliminary definitions, and prove two useful lemmas, including Lemma 2.3, which is a central technical tool for the rest of the paper. Section 3 gives a positive answer to Question 1.1 for rational and ruled surfaces. Sections 4 and 6 are the heart of the paper, giving a positive answer to Question 1.1 for a wide range of rationally connected threefolds: section 4 deals with Fano threefolds with Picard rank one, and section 6 with del Pezzo fibrations. In section 5, given a fibration $\pi\colon X \to \mathbb{P}^1$, we prove the existence of sections avoiding a subset of codimension at least two, which may be of independent interest. Finally, section 7 gives some applications of these results to integral points on families of curves and surfaces, including some classical cases of points integral with respect to a divisor.

2. Preliminaries

We first fix some notation and definitions. Let X be a projective algebraic variety, $Z \subset X$ a Zariski closed subset, both defined over a number field k. Let M_k be the set of places of k. The following definition is essentially Definition 1.4.3 in [Vo]:

Definition 2.1. *Let S be a finite set of places of k containing all the archimedean places. A subset $R \subset X(k) - Z(k)$ is called (Z,S)-*integralizable *if and only if there are global Weil functions $\lambda_{Z,v}$ and non-negative real numbers n_v such that $n_v = 0$ for all but finitely many places v for each v, and such that*

$$\lambda_{Z,v}(P) \leq n_v$$

for all $v \in M_k - S$ and $P \in R$.

*If the $\lambda_{Z,v}$ and n_v are fixed, we will say that a k-rational point P is (Z,S)-*integral *(or Z-*integral*, if S is understood) if and only if $\lambda_{Z,v}(P) \leq n_v$ for all $v \in M_k - S$.*

This definition is somewhat involved, and for the sake of brevity, we refer the reader to section 1.4 of [Vo] for a more detailed discussion.

We begin with a lemma.

Lemma 2.2. *Let $f\colon X \to Y$ be a birational morphism between irreducible varieties. Assume that f, X, and Y are all defined over the same number field k. If Question 1.1 has a positive answer for every algebraic subset $Z \subset Y$ of codimension at least two, then it has a positive answer for every algebraic subset $Z \subset X$ of codimension at least two.*

Proof: Let $Z \subset X$ be an algebraic subset of codimension at least two. Then $f(Z)$ is an algebraic subset of Y of codimension at least two, so by hypothesis the $f(Z)$-integral points of Y are potentially Zariski dense. But then the $f^{-1}(f(Z))$-integral points of X are potentially

Zariski dense as well, so *a fortiori* the Z-integral points of X are also potentially Zariski dense. ♣

The next lemma also appears as Theorem 3.1 in [MZ], but the proof given there is slightly different. The idea behind this lemma is to show that if a curve has infinitely many integral points on it, then deleting an arithmetic zero-cycle from it will either delete all the integral points, or else leave an infinite set of integral points. (If the curve C is projective, then "integral points" refers to rational points.)

In the statement of the lemma, C is the curve we're considering, and Z is the "locus at infinity" – that is, we assume that C has an infinite set of Z-integral points. We then delete a further set N which is assumed to intersect C in an arithmetic zero-cycle, and the assumption is that C contains at least one $(Z \cup N)$-integral point. Lemma 2.3 then says that C must still contain an infinite set of $(Z \cup N)$-integral points. In other words, Question 1.1 has a positive answer for curves, considered as arithmetic surfaces.

Lemma 2.3. *Let V be an algebraic variety defined over a number field k, and let $\mathcal{V}$ be a model of V over Spec$(\mathcal{O}_k)$. Let C be an irreducible curve on V, and let $\mathcal{C}$ be its closure in $\mathcal{V}$. Let Z and N be Zariski closed subsets of V, and let $\mathcal{Z}$ and $\mathcal{N}$ be their closures in $\mathcal{V}$, respectively. Let $L = Z \cup N$, and let S be a set of places of k that contains all the archimedean places of k.*

For every place v of k with $v \notin S$, let n_v be a non-negative real number. Assume that $n_v = 0$ for all but finitely many places v. Choose Weil functions $\lambda_{L,v}$ for each place v. Assume that there is a point $P \in C(k)$ satisfying

$$\lambda_{L,v}(P) \leq n_v$$

for every place $v \notin S$.

If $N \cap C = \emptyset$ and C contains an infinite set of (Z, S)-integral points, then there are infinitely many points $Q \in C(k)$ satisfying

$$\lambda_{L,v}(Q) \leq n_v$$

for every place $v \notin S$.

Proof: Since $C(k)$ is infinite, it follows that C must have geometric genus zero or one. The condition that C contain a dense set of Z-integral points implies that C must intersect Z in at most two places of C (places in the sense of points of the normalization of C), and that $C \cap Z = \emptyset$ if C has genus one.

We first assume that $C \cap Z = \emptyset$. Note that without loss of generality, we may assume that S is precisely the set of archimedean places of k, as increasing S only makes the lemma easier to prove.

Let S' be the set of places v of k such that either v is archimedean, or $\mathcal{N} \cap \mathcal{C}$ is supported on v. Note that S' is finite.

For each $v \notin S'$, $\lambda_{N,v}(Q) = 0$ for all $Q \in C$, so we may restrict our attention to $v \in S'$.

If v is finite with corresponding prime π of $\mathcal{O}_k$, then the condition $\lambda_{N,v}(P) < n_v$ depends only on the residue class of P modulo a suitable power of π. (See for example subsection 2.2.2 of [BG].) Thus, the collection of all $Q \in C(k)$ satisfying $\lambda_{N,v}(Q) \leq n_v$ for all finite v contains the set of points Q such that $Q \equiv P \pmod{M}$ for some suitable nonzero $M \in \mathcal{O}_k$.

There are now two cases: either the geometric genus of C is zero, or one.

If the geometric genus of C is zero, then the theorem follows immediately from the Weak Approximation Theorem for $\mathbb{P}^1$.

Weak Approximation does not hold for curves of genus one, however, so we must work a bit harder. The set B of points Q such that $Q \equiv P \pmod{M}$ for some nonzero $M \in \mathcal{O}_k$ contains the image on C of a coset of a finite index subgroup A of the Mordell-Weil group of the normalization $\tilde{C}$ of C over k. Since the set of rational points of C is infinite, the group A is infinite, and so we are done with the case $C \cap N = \emptyset$.

The only cases that remain are when C is a genus zero curve with either one or two places supported on Z. Let $\pi\colon \tilde{C} \to C$ be the normalization map over k. The set of points R of $\tilde{C}$ with $\pi(R) \notin Z$ are a principal homogeneous space for an arithmetic group ($\mathbb{G}_a$ if there is one place of C on Z, and $\mathbb{G}_m$ if there are two places), so by choosing a point R_0 on $\tilde{C}$, we can give the (π^*Z, S)-integral points of $\tilde{C}$ the structure of an arithmetic group G. The set B of points Q of C such that $Q \equiv P \pmod{N}$ for some nonzero $N \in \mathcal{O}_k$ – which, as before, is contained in the set of points Q satisfying $\lambda_{L,v}(Q) \leq n_v$ for all v not in S – contains the image on C of a coset of a finite index subgroup A of G, and is therefore infinite, as desired. ♣

We will apply Lemma 2.3 in the case where Z is empty.

3. Surfaces

If Lemma 2.3 gives a positive answer to Question 1.1 for curves, then the next natural question is to ask if it has a positive answer for surfaces. This is as yet unknown in general, but there are nevertheless a great many cases in which it is known. For example, Question 1.1 has a positive answer for every toric variety, by Corollary 4.2 in [HT]. In fact, we can prove much more.

Theorem 3.1. *Let X be a complex surface with negative Kodaira dimension, defined over a number field k. Then Question 1.1 has a positive answer for X.*

Proof: We begin with the following analogue of Lemma 2.3:

Lemma 3.2. *Let X be an algebraic surface, defined over a number field k, birational to $\mathbb{P}^2_k$ over k. Fix a model $\mathcal{X}$ for X over* $\mathrm{Spec}(\mathcal{O}_k)$*, and an effective arithmetic* 1*-cycle $\mathcal{Z}$ on $\mathcal{X}$, with $\mathcal{Z}$ defined over $\mathcal{O}_k$. Let S be a finite set of places of k including all the archimedean places. For each place $v \notin S$ of k, we fix a non-negative real number n_v and a Weil function $\lambda_{Z,v}$. If there is a $\mathcal{Z}$-integral point on X, then the set of $\mathcal{Z}$-integral points is Zariski dense.*

Proof: Let $f\colon X \dashrightarrow \mathbb{P}^2$ be a birational map defined over k. We may choose f so that there is a finite set of points $P_1 \ldots, P_n$ such that f restricts to a birational morphism from $X - \{P_1, \ldots, P_n\}$ to $\mathbb{P}^2$. In particular, this means that $f(\mathcal{Z})$ has dimension at most 1 on $\mathcal{X}$. We may also assume, *a fortiori*, that $\mathcal{Z} = f^{-1}(f(\mathcal{Z}))$. In that case, a point $P \in X$ is $\mathcal{Z}$-integral if and only if $f(P)$ is $f(\mathcal{Z})$-integral. It therefore suffices to show that the set of $f(\mathcal{Z})$-integral points in $\mathbb{P}^2$ is Zariski dense. This is well known – it follows, for example, from Corollary 4.2 of [HT], or from Theorem 4 of [Sh]. We include a proof here for completeness.

Let Q be a $\mathcal{Z}$-integral point in $\mathbb{P}^2$, guaranteed by the hypothesis. Then there is a line L through Q whose closure $\mathcal{L}$ over $\mathrm{Spec}(\mathbb{Z})$ meets $f(\mathcal{Z})$ in an arithmetic 0-cycle. By Lemma 2.3, the existence of one $f(\mathcal{Z})$-integral point Q on L implies the existence of infinitely many such points. For each such point Q', we may find a further line L', different from L, that passes through Q' and meets $f(\mathcal{Z})$ in an arithmetic 0-cycle. This means that the $f(\mathcal{Z})$-integral points on all the lines L' are also Zariski dense, so the $f(\mathcal{Z})$-integral points of $\mathbb{P}^2$ – and therefore also of X – are Zariski dense, as desired. ♣

Every rational surface is well known to be the blowup of some Hirzebruch surface or of the projective plane. Since all of those are toric varieties, Question 1.1 has a positive answer for them. Therefore, by Lemma 2.2, Question 1.1 has a positive answer for every rational surface.

If X is a ruled surface, then there is a fibration $f\colon X \to C$ for some smooth curve C. If C has genus at least two, then Question 1.1 has a vacuously positive answer for X, because the rational points on X are not potentially dense. If the genus of C is zero, then X is rational and Question 1.1 has a non-vacuously positive answer for X, as just noted.

Thus, assume that C has genus 1, and let Z be an algebraic subset of codimension at least two – that is, let Z be a finite set of points of X. By a finite extension of the field of definition k, we may assume that there is a Z-integral point P on X defined over k, and that C has an infinite number of k-rational points. By, for example, Theorem V.2.17.(c) of [Ha], there is a very ample divisor class V on X whose elements are sections of f, and therefore have infinitely many rational points. Let Y_1 be a curve in the class V that contains the point P, but does not intersect Z. By Lemma 2.3, Y_1 has a Zariski dense set of Z-integral points, and in particular, there are an infinite number of fibres F of f for which $Y_1 \cap F$ is a Z-integral point, and for which $F \cap Z = \emptyset$. By Lemma 2.3 again, this means that F has a dense set of Z-integral points, implying that the set of Z-integral points is dense, and that Question 1.1 has a positive answer for X.

Every surface of negative Kodaira dimension is the blowup of a rational or ruled surface. Thus, by Lemma 2.2, Theorem 3.1 is proven. ♣

For surfaces with non-negative Kodaira dimension, the situation is more complex, and indeed it is still not known which of these surfaces have a Zariski dense set of rational points, never mind integral ones. We will therefore move on to threefolds.

4. Fano threefolds

For the purposes of this paper, a Fano threefold is a smooth, three-dimensional algebraic variety X whose anticanonical sheaf $-K_X$ is ample. If X has Picard rank one – that is, if the Picard group of X is isomorphic to $\mathbb{Z}$ – then there is a unique ample generator H of the Picard group. The index of a Fano threefold is the unique integer r such that $-K_X = rH$. The main theorem of this section is the following:

Theorem 4.1. *Let X be a complex Fano threefold of Picard rank one and index at least two. Assume that X is defined over a number field k, and let Z be an algebraic subset of X of codimension at least two. If X is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,2,3)$, then we make the further assumption that Z does not contain the unique basepoint P of the square root of the anticanonical linear system. Then the Z-integral points of X are potentially Zariski dense.*

Proof: The proof relies crucially on the classification of Fano threefolds of Picard rank one, found (for example) in [IP]. Section 12.2 of [IP] gives the following list of Fano threefolds of Picard rank one and index at least two:

(a) $\mathbb{P}^3$
(b) A smooth quadric in $\mathbb{P}^4$.
(c) A smooth linear section of the Plücker-embedded Grassmannian $\mathrm{Gr}(2,5)$.
(d) A smooth intersection of two quadrics in $\mathbb{P}^5$.
(e) A smooth cubic hypersurface in $\mathbb{P}^4$.
(f) A double cover of $\mathbb{P}^3$, branched on a smooth quartic surface.
(g) A smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$.

Note that our list is in the reverse order of that in [IP], and in particular, item (g) is indeed the same as the unnamed threefold with $-K_X^3 = 8$ and $h^{1,2} = 21$.

We now proceed by cases. Case (a) is the easiest, as the answer to question 1.1 is well known to be positive for $\mathbb{P}^3$ – see for example [HT] or [Sh].

Case (b): X is a smooth quadric hypersurface in $\mathbb{P}^4$.

Let P be a k-rational point of $X - Z$, and let $\pi\colon X \to \mathbb{P}^3$ be the linear projection from P. Then $\pi(Z)$ is an algebraic subset of $\mathbb{P}^3$ of codimension at least two, and so the $\pi(Z)$-integral points are potentially Zariski dense. It therefore follows immediately that the Z-integral points of X are also potentially Zariski dense.

Case (c): X is a smooth linear section of the Plücker-embedded Grassmannian $\mathrm{Gr}(2,5)$.

X can be obtained by blowing up a smooth quadric threefold $Q \subset \mathbb{P}^4$ along a smooth rational curve of degree three, and then contracting the strict transform of a smooth quadric surface. If $\pi\colon Y \to Q$ is the blowup, and $\phi\colon Y \to X$ is the contraction, then a Z-integral point on X corresponds to a ϕ^*Z-integral point on Y. Any $\pi(\phi^*Z)$-integral point of Q pulls back to a ϕ^*Z-integral point of Y, so it suffices to show that the $\pi(\phi^*Z)$-integral points of Q are potentially Zariski dense.

The scheme $\pi(\phi^*Z)$ is contained in the union of a smooth quadric surface and a subset of Q of codimension at least two. It therefore suffices to prove that the Z-integral points of Q are potentially Zariski dense, where Z is the union of a smooth quadric surface S and a subset W of codimension at least two.

After a finite extension of the base field k, we may assume that there is a Z-integral point P on Q, and that the group $\mathcal{O}_k^*$ is infinite, where $\mathcal{O}_k^*$ is the group of units of the ring of integers $\mathcal{O}_k$ of k. Let T be a 2-plane containing P, but with $T \cap S$ finite, $T \cap Q$ irreducible and smooth at P, and $T \cap W = \emptyset$. Then $(T \cap Q) - (T \cap S)$ is a rational curve with at most two places deleted, and has a Z-integral point. Therefore, by Lemma 2.3, $T \cap Q$ contains infinitely many Z-integral points.

For each such point P', we can find another 2-plane T' such that $P' \in T'$, $T' \cap S$ finite, $T' \cap Q$ irreducible and smooth at P', $T' \cap W = \emptyset$, and $T' \neq T$. This, via Lemma 2.3, yields a set of Z-integral points whose Zariski closure Y has dimension at least 2. If $Y \neq Q$, then for each Z-integral point P_i'' on Y, we can find a 2-plane T_i'' such that $P_i'' \in T_i''$, $T_i'' \cap S$ finite, $T_i'' \cap Q$ irreducible and smooth at P_i'', $T_i'' \cap W = \emptyset$, and $T_i'' \cap Q \notin Y \cup T_1'' \cup \ldots \cup T_{i-1}''$. By Lemma 2.3, we obtain a set of Z-integral points that is Zariski dense in Q.

Case (d): X is a smooth intersection of two quadrics in $\mathbb{P}^5$.

After a finite extension of the base field k, we can choose a Z-integral point P that is not contained in Z, and such that the singular locus of the linear projection of X from P is not contained in the image of Z. Let $\pi_1 \colon X \to \mathbb{P}^4$ be the projection from P. Then $\pi_1(X)$ is a singular cubic threefold, and if P' is a singular point of $\pi_1(X)$ that is not contained in $\pi_1(Z)$, then the projection $\pi_2 \colon \pi_1(X) \to \mathbb{P}^3$ of $\pi_1(X)$ away from P' induces a birational map $\phi \colon X \to \mathbb{P}^3$ such that $\phi(Z)$ is an algebraic subset of $\mathbb{P}^3$ of codimension at least two. Since $\phi(X)$-integral points are well known to be potentially Zariski dense in $\mathbb{P}^3$, it follows that Z-integral points on X are also potentially Zariski dense, as desired.

Case (e): X is a smooth cubic threefold in $\mathbb{P}^4$.

Let ℓ be a line on X with $\ell \cap Z = \emptyset$, and let $\pi \colon Y \to X$ be the blowing up of X along ℓ. Then Y admits the structure of a conic bundle $\phi \colon Y \to \mathbb{P}^2$, where the exceptional divisor S of π is a rational surface and a double section of ϕ.

After a fixed extension of k, we may assume that ℓ, Y, π, and ϕ are all defined over k, and that S has a dense set of k-rational points, including a point P that is also Z-integral. By Lemma 3.2, this means that S has a Zariski dense set of Z-integral points as well.

The dimension of Z is at most one, so there is a dense set A of points of S such that for all $Q \in A$, the fibre of ϕ through Q does not meet Z. by Lemma 2.3, each such fibre has an infinite set of Z-integral points, and so the Z-integral points of X are dense.

Case (f): X is a double cover of $\mathbb{P}^3$ branched on a smooth quartic surface.

The threefold X is known to be unirational (see for example [IP], Example 10.1.3.(iii)), so we may extend the number field k to ensure that X has a Zariski dense set S of rational points. Let $\pi\colon X \to \mathbb{P}^3$ be the double cover. The set $\pi(S)$ is Zariski dense in $\mathbb{P}^3$, and by extending the field k again we may assume that at least one point P of $\pi(S)$ is $\pi(Z)$-integral.

Any line ℓ in $\mathbb{P}^3$ lifts to a curve of geometric genus at most one on X. The net N of lines through P induces an elliptic threefold structure (fibred over a rational surface) on a blowup $\tilde{X}$ of X. By Merel's Theorem on the uniform boundedness of torsion on elliptic curves (see [Me]), there is a proper Zariski closed subset G of $\tilde{X}$ which contains all the k-rational points of $\tilde{X}$ that are torsion points on their fibre. We further enlarge G to contain all the singular fibres of $\tilde{X}$. Let T be the complement of the image of G in $\mathbb{P}^3$, intersected with the set $\pi(S)$. Then T consists entirely of k-rational points Q of $\mathbb{P}^3$ whose preimages on X are also k-rational, and such that the elliptic curve lying over the line joining the point Q to P has positive rank. (The point P is viewed as the identity element.)

Since Z has codimension at least two, there is a Zariski dense set of points Q in T such that the line ℓ joining P to Q is disjoint from Z. In each such case, the elliptic curve E lying over ℓ has positive Mordell-Weil rank (because Q is non-torsion with respect to P), and so by Lemma 2.3, E contains an infinite set of Z-integral points. Since the set of such E is Zariski dense, the theorem follows.

Case (g): X is a smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$, and the subset Z does not contain the unique basepoint P of the square root of the anticanonical linear system.

In this case, X is a double cover of the cone V in $\mathbb{P}^6$ over the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$; denote the cover by $\pi\colon X \to V$. Note that P is the preimage $P = \pi^{-1}(v)$ of the vertex v of the cone V.

Blowing up v on V yields a smooth threefold V^*, which admits the structure of a $\mathbb{P}^1$-bundle $f\colon V^* \to \mathbb{P}^2$ over $\mathbb{P}^2$, where the fibres of the bundle are the strict transforms of the lines of the ruling of V. The corresponding blowup of X yields a biregular map $g\colon X^* \to X$ and a double cover $\pi^*\colon X^* \to V^*$, where X^* inherits the structure of an elliptic fibration over $\mathbb{P}^2$ via $\phi = f \circ \pi^*$.

Since $P \notin Z$, it follows that $Z^* = g^{-1}(Z)$ is at least codimension two as a subset of X^*. In [BT], the authors show that there is a two-dimensional family of double sections of ϕ that are singular, but birational to $K3$ surfaces. After a possible finite field extension, we may assume that one of those double sections, which we will call S, satisfies the following properties:

- S intersects Z properly.
- S contains a singular point s which is Z^*-integral.
- S contains a Zariski dense set of rational points.

To see that such a choice is possible, note that [BT] proves the Zariski density of the rational points, and allows for a two-dimensional linear system full of such double sections S. (The extra field extension is necessary for the existence of the Z^*-integral singular point.)

Given such an S, we blow up the singular locus with $h\colon S^* \to S$ to obtain an elliptically fibred, smooth $K3$ surface S^*. The exceptional divisor over s is a (-2)-curve on S^* with a dense set of rational points, each of which is $h^{-1}(Z^*)$-integral. Thus, every smooth elliptic curve in the elliptic fibration on S^* contains at least one $h^{-1}(Z^*)$-integral point. The density of rational points on S^* implies that there are infinitely many such curves with positive Mordell-Weil rank. Therefore, since $Z^* \cap S$ is of codimension at least two, we conclude by Lemma 2.3 that the set of $h^{-1}(Z^*)$-integral points on S^* is Zariski dense, and therefore that the Z^*-integral points on S are also Zariski dense on S.

For any Z^*-integral point x on X^*, Lemma 2.3 again shows that the Z^*-integral points are Zariski dense on the fibre of ϕ through x, provided that its Mordell-Weil rank is positive. Since [BT] proves that the set of rational points on X^* are Zariski dense, there is a Zariski dense set of fibres with positive Mordell-Weil rank. Therefore, the set of Z^*-integral points on X^* is Zariski dense. This immediately implies that the set of Z-integral points on X is Zariski dense, as desired. ♣

5. Sections avoiding given subsets

In this section we prove a lemma guaranteeing the existence of a section of a rationally connected fibration over a curve, such that the section avoids (respectively fails to be contained in) a given subset of codimension ≥ 2 (respectively ≥ 1). The result is well-known to experts on families of curves on varieties, but we include a proof for lack of a reference. We begin by recalling background material.

Let X be a smooth projective variety, and set $n = \dim(X)$.

Recall that a *rational curve* in X is a nonconstant map $f\colon \mathbb{P}^1 \longrightarrow X$. The rational curve is said to be free if $f^*T_X = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$, with each $a_i \geq 0$. Fix an ample line bundle L on X. Then for each $d \geq 1$ there is a quasi-projective variety $\mathrm{Hom}(\mathbb{P}^1, X)_d$ parameterizing maps $f\colon \mathbb{P}^1 \longrightarrow X$ such that $\deg(f^*L) = d$. (This is a special case of the general construction of [Ko, 1.10] constructing parameter spaces $\mathrm{Hom}(Y, X)$ for any projective varieties Y and X. Any morphism $Y \longrightarrow X$ can be identified with its graph, a subset of $Y \times X$, and the spaces $\mathrm{Hom}(Y, X)$ are then realized as the open subscheme of $\mathrm{Hilb}(Y \times X)$ parameterizing such graphs. The restriction $\deg(f^*L) = d$ is used to fix the Hilbert polynomial of the graph.)

For a map $f\colon \mathbb{P}^1 \longrightarrow X$, with $\deg(f^*L) = d$, we denote by $[f]$ the corresponding point of $\mathrm{Hom}(\mathbb{P}^1, X)_d$. One also has an *evaluation map*

$$\begin{array}{cccc} \mathrm{ev} & : & \mathrm{Hom}(\mathbb{P}^1, X)_d \times \mathbb{P}^1 & \longrightarrow & X \\ & & ([f], p) & \longmapsto & f(p). \end{array}$$

Let $\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ$ denote the subset of $\mathrm{Hom}(\mathbb{P}^1, X)_d$ consisting of those $[f]$ such that f is free. By [Ko, II.3.5.4, p. 115], $\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ$ is an open subset of $\mathrm{Hom}(\mathbb{P}^1, X)_d$, and the evaluation map

$$\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ \times \mathbb{P}^1 \xrightarrow{\ \mathrm{ev}\ } X$$

is smooth. (Thus $\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ$ is also smooth, although one can see this last point directly by computing the tangent space to the Hilbert scheme).

Lemma 5.1. *(a) Let X be a smooth irreducible projective variety defined over an algebraically closed field of characteristic zero, $\pi\colon X \longrightarrow \mathbb{P}^1$ a surjective map whose general fibre is rationally connected, $Z \subset X$ a subvariety of codimension ≥ 2, and $T \subset X$ a subvariety of codimension ≥ 1. Then there exists a section of π which is not contained in T, and which does not meet Z. (b) If X, and π, Z, and T are defined over a field k of characteristic zero, and if the general fibre of π over $\overline{k}$ is rationally connected, then there exists such a section defined over a finite extension k' of k.*

Proof: We first prove (a). By [GHS, Theorem 1.1] there is a map $g\colon \mathbb{P}^1 \longrightarrow X$ which is a section of π. Furthermore, by [KMM, 2.13], given that such a section exists, and given any point q on a smooth fibre of π, there exists a curve $f'\colon \mathbb{P}^1 \longrightarrow X$ which is a free curve, a section of π, and passes through q (i..e, so that q is in the image of f').

Choose any point q in a smooth fibre, and not in Z or T, and let f' be a free curve and section passing through q provided by those theorems. Set $d = \deg((f')^*L)$, and let V be the irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ$ containing $[f']$. We consider the diagram

$$\begin{array}{ccc} V \times \mathbb{P}^1 & \xrightarrow{\text{ev}} & X \\ \Big\downarrow{\scriptstyle p_1} & & \\ V & & \end{array} .$$

The property that a rational curve f is a section of π is equivalent to $\deg(f^*\pi^*\mathcal{O}_{\mathbb{P}^1}(1)) = 1$, i.e., that the degree of $\text{ev}^*\,\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibre $p_1^{-1}([f])$ is 1. Since p_1 is flat, the degree of $\text{ev}^*\,\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is constant on the fibres of p_1 and it follows that every $[f] \in V$ is also a section of π.

The map p_1 is proper, and by [Ko, II.3.5.4.2, p. 115], ev is smooth. The set of $[f] \in V$ such that $f(\mathbb{P}^1)$ is contained in T is the locus where the map $\text{ev}^{-1}(T) \xrightarrow{p_1} V$ has 1-dimensional fibres. By upper semicontinuity of fibre dimension this locus is a closed subset of V. Let U' be its complement. The set U' is nonempty since $[f'] \in U'$. Every point $[f] \in U'$ is now a section of π not contained in T. To prove part (a) we just need to find such an $[f]$ so that $f(\mathbb{P}^1) \cap Z = \emptyset$.

Set $N = \dim(V)$. Since Z is of codimension ≥ 2, and ev smooth, $\text{ev}^{-1}(Z)$ also has codimension ≥ 2, and hence has dimension at most $N+1-2 = N-1$. Thus $p_1(\text{ev}^{-1}(Z))$ has dimension $\leq N-1$ and so is a proper subset of V. Let U'' be its complement. Any $[f] \in U''$ satisfies $f(\mathbb{P}^1) \cap Z = \emptyset$. Since V is irreducible, $U := U' \cap U'' \neq \emptyset$, proving (a).

To see (b) we first note that if X is defined over k, then we can choose an ample L defined over k, and then $\text{Hom}(\mathbb{P}^1, X)_d$ is also defined over k for each $d \in \mathbb{N}$. (As above, one starts with the Hilbert scheme $\text{Hilb}(\mathbb{P}^1 \times X)$ and restricts to the open subset which are the graphs of morphisms. The Hilbert scheme and the condition of being a graph can be expressed over k.) The open condition that a morphism f is free is similarly defined over k, as is the condition that f is a section of π (this being again a condition on the degree of $\text{ev}^*\,\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibres of p_1).

If T and Z are defined over k, then the closed locus where the fibre dimension of $\text{ev}^{-1}(T) \longrightarrow \text{Hom}(\mathbb{P}^1, X)_d^\circ$ is 1 is defined over k, and so is the closed subset $p_1(\text{ev}^{-1}(Z))$. Thus, the intersection of their complements is also defined over k. For each $d \in \mathbb{N}$ we let $U_d \subseteq \text{Hom}(\mathbb{P}^1, X)_d$ be the intersection of the complements, along with the intersection with the open conditions of the maps being free, and being a section of π.

By part (a), over $\overline{k}$, there is some d for which the corresponding U_d is nonempty. But, if U_d is nonempty after base extension, it was nonempty to begin with. Since $\text{Hom}(\mathbb{P}^1, X)_d$ is of finite type, the residue field of any closed point is finite over k. Thus taking any closed point $[f] \in U_d$, and letting k' be its residue field, we obtain a section $f\colon \mathbb{P}^1 \longrightarrow X$ defined over k', avoiding Z, and not contained in T. ♣

Using an idea from [GHS] due to de Jong, one can extend Lemma 5.1 to the case where the base curve has arbitrary genus. We will not need this extension, but record the statement and the idea of its proof.

Corollary 5.2. *(a) Let X be a smooth irreducible projective variety defined over an algebraically closed field of characteristic zero, $\pi\colon X \longrightarrow C$ a surjective map to a smooth curve C such that the general fibre of π is rationally connected, $Z \subset X$ a subvariety of codimension ≥ 2, and $T \subset X$ a subvariety of codimension ≥ 1. Then there exists a section of π which is not contained in T, and which does not meet Z. (b) If X, and π, Z, and T are defined over a field k of characteristic zero, and if the general fibre of π over $\overline{k}$ is rationally connected, then there exists such a section defined over a finite extension k' of k.*

Proof: We repeat the argument of de Jong from [GHS, §3.2]. To prove (a), given $\pi\colon X \longrightarrow C$ choose any finite map $g\colon C \longrightarrow \mathbb{P}^1$, and then form the "norm" of X. This is a variety and map $\varphi\colon Y \longrightarrow \mathbb{P}^1$ (well defined up to birational equivalence) whose fibre over a general point $p \in \mathbb{P}^1$ is the product $\prod_{q\in g^{-1}(p)} \pi^{-1}(q)$. The utility of the norm construction is that sections of φ give sections of π. Given a section σ of φ, for each $p \in \mathbb{P}^1$, σ gives a point of $\prod_{q\in g^{-1}(p)} \pi^{-1}(q)$, and thus for each point $q \in C$, setting $p = g(q)$, σ gives a point in the fibre $\pi^{-1}(q)$.

To ensure that the resulting section of π misses Z and is not contained in T, we define appropriate subsets of Y. Let $\tilde{Z} \subset Y$ be the subset

$$\tilde{Z} = \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{at least one of the coordinates of } y \in \\ \varphi^{-1}(\varphi(y)) = \prod_{q\in g^{-1}(\varphi(y))} \pi^{-1}(q) \text{ is in } Z \end{array} \right\}$$

and similarly define $\tilde{T}$.

Sections σ of φ which do not meet $\tilde{Z}$ and are not contained in $\tilde{T}$ induce sections of π similarly missing Z and not contained in T. The codimensions of $\tilde{Z}$ in Y is equal to the codimension of Z in X, and similarly $\operatorname{codim}(\tilde{T}, Y) = \operatorname{codim}(T, X)$.

Since the product of rationally connected varieties is rationally connected, the general fibre of Y is rationally connected, and so we can apply Lemma 5.1(a), proving (a) of the corollary.

To prove (b), supposing everything defined over k, if we choose our map $g\colon C \longrightarrow \mathbb{P}^1$ to be defined over k, then so are Y, $\tilde{Z}$, and $\tilde{T}$. Thus applying Lemma 5.1(b), we obtain a section of φ defined over a finite extension k' missing $\tilde{Z}$ and not contained in $\tilde{T}$. This then induces a section of π, also defined over k', with the desired properties. ♣

6. Del Pezzo fibrations

in this section, we prove the potential density of integral points for del Pezzo fibrations, provided that the degree of the (generic) del Pezzo surface is at least three.

Let $\pi\colon X \to Y$ be a morphism, where X is a smooth, rationally connected, projective threefold, and Y is a smooth curve. Since X is rationally connected, Y must be isomorphic to $\mathbb{P}^1$ over k. (This may require a finite extension of k.) We further assume that a general fibre of π is a del Pezzo surface.

Choose models $\mathcal{X}$ and $\mathcal{Y}$ for X and Y, respectively, over $\mathcal{O}_k$, and extend π to a rational map from $\mathcal{X}$ to $\mathcal{Y}$. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed subscheme of codimension at least two. We will show that in many cases, the $\mathcal{Z}$-integral points of $\mathcal{X}$ are potentially Zariski dense.

Theorem 6.1. *Let k be a number field. Let X be a smooth threefold with a map $\pi\colon X \to \mathbb{P}^1$ whose generic fibre is a del Pezzo surface of degree at least three, all defined over k. Let $Z \subset X$ be an algebraic subset of codimension at least two. Then the Z-integral points of X are potentially Zariski dense.*

Proof: Let T be the union of the (-1)-curves in the fibres of π. Applying Lemma 5.1, after at most a finite field extension – which we continue to call k – we obtain a k-rational section $\sigma\colon \mathbb{P}^1 \to X$ of π whose image is a smooth rational curve $C \subset X$, and disjoint from Z, and meeting T in only finitely many points (i.e., only finitely many points of C are contained in (-1)-curves of the fibres of π). Furthermore, after blowing up, we may decrease the degree of the generic fibre of π to three without changing the hypothesis or conclusion of the Theorem 6.1. (We choose the blowup locus to be disjoint from Z.) Let $S \subset \mathbb{P}^1(k)$ be the finite subset of points p where either $\pi^{-1}(p)$ contains a 1-dimensional component of Z, or $\pi^{-1}(p)$ intersects C in a point on a (-1)-curve of the fibre.

Theorem 6.1 then follows by applying the following lemma to the fibres $\pi^{-1}(p)$, with $p \in \mathbb{P}^1(k) \setminus S$. (Note that Lemma 6.2 is not implied by Lemma 3.2 because a del Pezzo surface need not be birational to $\mathbb{P}^2$ over k.)

Lemma 6.2. *Let V be a del Pezzo surface of degree three defined over a number field k, and let $\mathcal{V}$ be a model for V over* $\mathrm{Spec}(\mathcal{O}_k)$. *Let $\mathcal{Z} \subset \mathcal{V}$ be an algebraic subset of codimension at least two. Assume that there is a k-rational point $P \in V(k)$ that is $\mathcal{Z}$-integral, and that does not lie on a* (-1)*-curve of V. Then the $\mathcal{Z}$-integral points are Zariski dense.*

Proof of lemma: A general member of the linear system $|-K_V|$ is a smooth curve of genus one, and $|-K_V|$ is basepoint free because V is del Pezzo. Consider the linear subsystem of $|-K_V|$ consisting of curves containing P. It has dimension three, so we can choose a pencil H of curves defined over k such that every curve in H contains P, a general curve in H is smooth, and the base locus consists of three points $\{P, Q, R\}$, none of which lie in Z. (Note that $P \notin Z$ trivially.)

Since H is defined over k, so is the triple $\{P, Q, R\}$, and we may therefore blow it up to obtain a surface $\tilde{V}$ defined over k, with a morphism $\psi\colon \tilde{V} \to V$ whose fibres are precisely the (strict transforms of) the curves in H. The morphism ψ makes $\tilde{V}$ into an elliptic surface, with a section $\mathcal{O}$ given by the exceptional curve lying over P. Note that $\mathcal{O}$ is disjoint from Z. Since ψ has a section, it has no multiple fibres, so we may enlarge Z to contain all the singular points of fibres of ψ.

The class $-K_V$ embeds V in $\mathbb{P}^3$ as a smooth cubic surface. We may therefore consider the curve T defined by the intersection of the tangent plane T_P with the embedded surface V. Note that T is irreducible because P does not lie on any (-1)-curves, so T is an irreducible plane cubic curve. Moreover, T is singular at P, so it has geometric genus zero. Indeed, T is birational to $\mathbb{P}^1$ over k via projection from P in the plane T_P, so T has a dense set of rational points.

For each rational point A of T, the intersection of T_A with the embedded surface V is again a cubic curve with a singularity at A, albeit possibly reducible. At most finitely many A correspond to reducible curves in this way (there are only finitely many intersections of T with lines), so there are infinitely many A whose tangent curves are birational to $\mathbb{P}^1$ over k, and therefore have a dense set of rational points. We therefore deduce that the rational points of V are Zariski dense.

This means that there is a dense set of k-rational points on V each lying on a smooth fibre of ψ and having infinite order in that fibre. (To see this, note that by a theorem of Merel ([Me]), there is a positive integer N such that for any elliptic curve defined over k, and any k-rational point A of finite order, the order of A divides N. Therefore, the set of k-rational points of finite order in their fibre is not Zariski dense in V.) In particular, there are an infinite number of genus one curves on V that contain an infinite set of k-rational points, one of which is the Z-integral point P. By Lemma 2.3, each of those curves contains an infinite set of Z-integral points. We conclude that the Z-integral points on V are dense. ♣

We now finish the proof of Theorem 6.1. The curve C is disjoint from Z over the generic fibre, so after a further finite extension of k – which we stubbornly persist in calling k – we may assume that C contains a Z-integral point. To see this, note that over $\mathrm{Spec}(\mathcal{O}_k)$, $\mathcal{N} = \mathcal{C} \cap \mathcal{Z}$ is an arithmetic zero-cycle supported on finitely many places of k. After a suitably chosen finite extension of k, we may assume that for every place v of k, there is a point p_v of $\mathcal{C}$ lying over v that does not lie in the support of $\mathcal{N}$. By the Chinese Remainder theorem – since C is a rational curve – there is some k-rational point P of C such that for all v over which $\mathcal{N}$ is supported, $P \equiv p_v \mod v$. This P is the Z-integral point that we seek.

Since C is a rational curve, Lemma 2.3 implies that C contains an infinite number of Z-integral points. For each such point, the corresponding fibre, by Lemma 3.2, contains a dense set of Z-integral points. It therefore follows that the Z-integral points of V are Zariski dense, as desired. ♣

7. Application to integral points in families

We can apply the theorems of the previous sections to families of curves and surfaces on surfaces and threefolds, to get results about integral points in the classical sense – that is, integral points with respect to a divisor. The proof of the following theorem is trivial:

Theorem 7.1. *Let X be a smooth, projective variety of dimension n, defined over a number field k, and let $\mathcal{P}$ be an $(n-m)$-dimensional linear system of m-dimensional cycles on X whose base locus Z is of pure dimension $m-1$. (In other words, elements of a basis for $\mathcal{P}$ intersect properly.) If the Z-integral points of X are Zariski dense, then there is a Zariski dense set of cycles in $\mathcal{P}$ with at least one Z-integral point defined over k.*

Note that Z is a divisor on each of the cycles in $\mathcal{P}$, so that a Z-integral point on a surface S in the family is an integral point in the classical sense. Indeed, if the divisors in $\mathcal{P}$ are ample on X, then the base locus Z is the intersection of any $m+1$ distinct cycles in $\mathcal{P}$, so Z is an ample divisor on any smooth, irreducible cycle in the system.

The results of this paper show that the hypotheses of Theorem 7.1 are satisfied when X is any rational or ruled surface, or any rationally connected threefold of a type considered in sections 4 or 5.

In particular, though, the case $m = 1$, in which the cycles in $\mathcal{P}$ are curves, is particularly interesting. For any subset $Z' \subset Z$, any Z-integral point is automatically also Z'-integral. Therefore, when $m = 1$, the reduced induced subscheme of Z is a finite set of points, so we may assume that Z is a single point, or even a different single point of Z for every curve in $\mathcal{P}$, and the conclusion of Theorem 7.1 still holds. This is significant because classically, one is often interested in integral points on elliptic or hyperelliptic curves where the divisor "at infinity" is one or two points.

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