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A reduction of the Batyrev-Manin Conjecture for Kummer Surfaces

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Abstract

Let V be a $K3$ surface defined over a number field k. The Batyrev-Manin conjecture for V states that for every nonempty open subset U of V, there exists a finite set Z_U of accumulating rational curves such that the density of rational points on $U - Z_U$ is strictly less than the density of rational points on Z_U. Thus, the set of rational points of V conjecturally admits a stratification corresponding to the sets Z_U for successively smaller sets U.

In this paper, in the case that V is a Kummer surface, we prove that the Batyrev-Manin conjecture for V can be reduced to the Batyrev-Manin conjecture for V modulo the endomorphisms of V induced by multiplication by m on the associated abelian surface A. As an application, we use this to show that given some restrictions on A, the set of rational points of V which lie on rational curves whose preimages have geometric genus 2 admits a stratification of Batyrev-Manin type.

Keywords: Rational points, Batyrev-Manin conjecture, Kummer surface, rational curve, abelian surface, height
MSC: 11G35, 14G05

1 Introduction

Let V be an algebraic variety defined over a number field k, and let D be an ample divisor on V. Choose a (multiplicative) height function H_D on V corresponding to D, and let S be any subset of V. We define the counting function for S by:

$$N_{S,D}(B) = \#\{P \in S(k) \mid H_D(P) < B\}$$

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Batyrev and Manin [2] have made a series of conjectures about the behaviour of $N_{S,D}$. In the case that V is a $K3$ surface, their conjecture can be stated as follows:

Conjecture 1.1 *Let $\epsilon > 0$ be any real number. Then there is a non-empty Zariski open subset $U(\epsilon) \subset V$ such that*

$$N_{U(\epsilon),D}(B) = O(B^{\epsilon})$$

The complement $Z(\epsilon)$ of $U(\epsilon)$ can clearly be taken to be a finite union of rational curves, since it is well known that if C is a curve of genus at least 1 defined over a number field k, then its counting function grows at most like a power of the logarithm:

$$N_{C,D}(B) = O((\log B)^{\ell})$$

where ℓ is some fixed positive integer, and D is any ample divisor on C. Conversely, any rational curve C which is split over k (meaning that C is defined over k and has at least one k-rational closed point) satisfies:

$$N_{C,D}(B) \gg\ll B^{2/d}$$

where d is the D-degree of C. (In fact, the constants involved in the preceding expression can sometimes be calculated quite precisely – see for example [11].) In particular, this means that every k-split rational curve on V must be contained in $Z(\epsilon)$ for some $\epsilon > 0$.

Thus, as $\epsilon \to 0$, we conjecturally obtain a stratification of "most" of the k-rational points of V via the rational curves of minimal degree on which they lie. In fact, there is a conjecture of Bogomolov (see the remarks after Theorem 3.5 of [2]) which states that every algebraic point on a $K3$ surface lies on a rational curve defined over $\overline{k}$. Thus, disregarding the issue of the field of definition of a rational curve, Bogomolov's conjecture suggests that in one's analysis of the set of rational points of a $K3$ surface V, one may confine one's attention to the set of rational points which lie on rational curves. Even without appealing to Bogomolov's conjecture, Conjecture 1.1 already implies that the set S of k-rational points of V which do not lie on any rational curve is sparse in the sense that for any $\epsilon > 0$, we have:

$$N_{S,D}(B) = O(B^{\epsilon})$$

To my knowledge, this conjecture has not been verified for any $K3$ surface V. However, some progress has been made. For instance, King and Todorov [6] construct the first set in the stratification for a certain class of Kummer surfaces, and McKinnon [7] constructs the first few layers of the stratification for a wider class of $K3$ surfaces, both using geometric techniques.

Billard [1] computes the counting function for certain other $K3$ surfaces, but the more significant contribution of his paper is in the other direction. Namely, the preceding results consider the set of all rational points, and construct part of their stratification, while Billard considers a certain subset of rational points, and shows that it admits a stratification of the form predicted by Conjecture 1.1. In the parlance of §2, we say that Billard's subset is a stratified set.

More precisely, Billard starts with a single rational curve, and uses a carefully chosen infinite group of automorphisms to move it around the $K3$ surface. Using the canonical height introduced by Silverman in [12], Billard controls the heights on the images of the original curve in terms of the height on the original curve.

This technique is similar to that employed by Sato [10] for Kummer surfaces whose associated abelian surface A is isogenous to a product $E \times E$ of an elliptic curve with itself. Sato proves that the set of rational points lying in the image of some proper abelian subvariety of A is a stratified set. (In fact, Sato proves slightly more than this, but his other results follow easily from this one.)

As far as the author is aware, the most general result in this direction was obtained by Call [3], though he did not advertise it as such. Call proves that on a family of abelian varieties, the set of geometric points (that is, points lying on sections) is a stratified set, in the sense of §2, provided that the base of the family is a rational curve. (In his paper, Call purports only to compute the counting function for geometric points, but a careful examination of his proof shows that he in fact constructs the entire stratification, and reports only on the top layer.)

The purpose of this paper is to try to bridge the "top-down" and "bottom-up" approaches of the preceding papers, by treating the stratified set as the basic operand. We will confine our attention to the case of a Kummer surface, and to the rational endomorphisms $\widetilde{[m]}$ induced by multiplication by $m \in \mathbb{Z}$ on the associated abelian surface. We prove (Theorem 3.3) that if you start with a stratified set S, then the union of the sets $\widetilde{[m]}(S)$ is also a stratified set, and similarly for sparse sets. This effectively reduces Conjecture 1.1 on such Kummer surfaces to the equivalent conjecture for the smaller set of points which are "indivisible".

In particular, we prove some elementary results on stratified sets in §2, and in §3 prove an analogue of Sato's results for points in the image of a curve of genus 2 on the abelian surface (Corollary 3.4). The main theorems are proven in §4.

2 Stratified Sets

Let k be a number field. Let X/k be an algebraic surface. We recall the definition of a big divisor.

Definition 2.1 *A divisor D on a smooth projective variety V is called* big *if and only if for some positive integer n, nD is linearly equivalent to $A + E$ for some ample divisor A and some effective divisor E.*

Let H_D be a height function on X with respect to some big divisor D. For any subset $S \subset X$, we define the counting function $N_{S,D}(B)$ of S with respect to D by:

$$N_{S,D}(B) = \#\{P \in S(k) \mid H_D(P) < B\}$$

If the big divisor D has been fixed, we will drop D from the notation.

Let $S \subset X$ be a union of rational curves defined and split over k. We say that S is a stratified set (over k), if for every $\epsilon > 0$, there is a finite union $C(\epsilon) \subset S$ and a positive real constant c such that for all real numbers B:

$$\#\{P \in (S - C(\epsilon))(k) \mid H_D(P) < B\} \leq cB^\epsilon$$

Note that if D is not ample, then every $C(\epsilon)$ must contain all the curves C of S satisfying $C.D \leq 0$.

We now prove some elementary results about stratified sets. These results are well known, but are included here for clarity of presentation.

Theorem 2.2 *Let S be a stratified set with respect to a big divisor D, and let d be a positive integer. Then there are only finitely many rational curves $C \subset S$ of D-degree d. Moreover, the set $C(\epsilon)$ may be taken to be the set of rational curves in S of D-degree at most $2/\epsilon$.*

Proof: By Schanuel's Theorem [11], a split rational curve of degree d contains $cB^{2/d} + O(B^{2/d-\epsilon})$ rational points of height at most B, where c is a constant which Schanuel calculates explicitly. Thus, for $\epsilon < 2/d$, the set $C(\epsilon)$ must contain all the rational curves in S of degree d. Since $C(\epsilon)$ is a finite union of curves, the finiteness result follows.

For the other claim, note that one may always enlarge $C(\epsilon)$, so it suffices to show that one need not include any curve of degree larger than $2/\epsilon$ in $C(\epsilon)$. If C is a rational curve of degree larger than $2/\epsilon$, then Schanuel's Theorem says that it contains at most $O(B^\epsilon)$ rational points. Thus, by deleting it from $C(\epsilon)$, one does not increase the asymptotic number of rational points in $(S-C(\epsilon))(k)$, which is already $O(B^\epsilon)$. ♣

Thus, a stratified set is a union of rational curves such that the set of curves of bounded degree has a higher density of rational points than all the other curves put together.

Theorem 2.3 *The finite union of stratified sets is a stratified set.*

Proof: This is clear. ♣

Theorem 2.4 *The definition of a stratified set is independent of the choice of big divisor D.*

Proof: Assume S is a stratified set with respect to a big divisor D on X, and let D' be any other big divisor. Then we can find positive real constants c_1 and c_2 such that $c_1D - D'$ and $c_2D' - D$ are big. By the height machine [13], this means that we can find a dense open subset $U \subset X$ such that for all points $P \in U(k)$, we have:

$$H_D(P) \leq H_{D'}^{c_2}(P) \quad \text{and} \quad H_{D'}(P) \leq H_D^{c_1}(P)$$

Fix a positive real number ϵ. Then there is a finite union C of rational curves and a positive constant c such that for all real numbers B:

$$\#\{P \in (S - C)(k) \mid H_D(P) < B\} \leq cB^{\epsilon/c_2}$$

Now fix a real number B, and let $P \in (S - C)(k)$ be any point such that $H_{D'}(P) \leq B$. Then $H_D(P) \leq B^{c_2}$, so by the previous inequality there are at most $c(B^{c_2})^{\epsilon/c_2} = cB^\epsilon$ choices for P. Thus:

$$\#\{P \in (S - C)(k) \mid H_{D'}(P) < B\} \leq cB^\epsilon$$

as desired. ♣

It is not clear to the author if the definition of a stratified set is independent of the field k. However, this dependence is not relevant to the paper, and so we will generally not note the dependence on k.

3 Kummer Surfaces

Let A be an abelian surface, and let $i\colon A \to A$ be multiplication by -1. Then the quotient by i is a singular $K3$ surface S, with sixteen singular points, corresponding to the sixteen 2-torsion points of A, which are the fixed points of the involution i. Blowing up these sixteen points, we obtain a nonsingular $K3$ surface X, called the Kummer surface associated to A.

Let D be a big divisor on X, and let H_D be the associated (multiplicative) height function. We say that a set S of k-rational points on X is sparse if and only if we have:

$$N_S(B) = \#\{P \in S | H_D(P) \leq B\} = O(B^\epsilon)$$

for every ϵ. That is, S is sparse if and only if its counting function grows more slowly than the counting function of every (split) rational curve. As for stratified sets, the notion of sparsity is independent of the choice of big divisor D.

The Batyrev-Manin conjecture [2] for X can be stated in the following form:

Conjecture 3.1 (Batyrev-Manin) *The set $X(k)$ of rational points on X is the union of a stratified set and a sparse set.*

Batyrev and Manin conjectured this to hold for an arbitrary $K3$ surface. The purpose of this paper is to show that Conjecture 3.1 is true for $X(k)$ if and only if it is true for $X(k)$ modulo the endomorphisms of the abelian surface A.

More precisely, let $\phi\colon A \to A$ be any endomorphism. Then ϕ induces a rational map $\tilde{\phi}\colon X \to X$, since ϕ respects the inversion operation on A. For any subsets $S \subset X$ and $R \subset \mathrm{End}_k(A)$, define the set:

$$E_R(S) = \bigcup_{\phi \in R} \tilde{\phi}(S)$$

We prove the following two theorems:

Theorem 3.2 *Let $R = \mathbb{Z}$. Let S be a sparse set on X. Then the set:*

$$E_R(S) = \bigcup_{m \in \mathbb{Z}} \widetilde{[m]}(S)$$

is a sparse set on X.

Theorem 3.3 *Let $R = \mathbb{Z}$. Let S be a stratified set on X. Then the set:*

$$E_R(S) = \bigcup_{m \in \mathbb{Z}} \widetilde{[m]}(S)$$

is a stratified set on X.

Note that these theorems have consequences beyond a reduction of the Batyrev-Manin Conjecture. For instance, we have the following result:

Corollary 3.4 *Assume A is principally polarised and has End$(A) \cong \mathbb{Z}$. Let V be the set of all curves on A which are:*

1. *defined over k*
2. *of geometric genus two (that is, their normalisation has genus 2)*
3. *such that the inversion map i on A induces the hyperelliptic involution on V.*

(These are precisely the genus two curves whose images in X are rational curves.) Then the image on X of the union of the curves in V is a stratified set.

Proof: By Theorem 3.3, it suffices to show that there are only a finite number of such genus two curves on A, modulo the action of the endomorphism ring of A. Thus, assume that C is a smooth curve of genus two equipped with a non-constant map $\phi\colon C \to A$, birational onto its image, such that $\phi(C)$ is in V. Let $\pi\colon C \to J(C)$ be the embedding of C in its Jacobian $J(C)$. Then we have a canonical map:

$$\psi\colon J(C) \to A$$

which commutes with ϕ – that is, $\phi = \psi \circ \pi$.

If C' is the translate of $\phi(C)$ by a point $P \in A(k)$, then $C' \notin V$ unless P is a 2-torsion point. Thus, we may reduce the problem by translations as well, and assume that the map ψ is a homomorphism of abelian groups. This immediately implies that ψ is an isogeny.

Let $\phi' \colon C \to A$ be any other birational map, with associated isogeny ψ'. Identify both $J(C)$ and A with their duals, and let $\hat{\psi}$ and $\hat{\psi}'$ be the dual maps. Then we have:

$$\hat{\psi}\psi = \alpha$$

for some $\alpha \in \mathrm{End}(J(C))$. Since $J(C)$ is isogenous to A, we know that $\mathrm{End}(J(C)) = \mathbb{Z}$, so we get:

$$\hat{\psi}\psi = [m]$$

where $[m]$ denotes multiplication by the nonzero integer m. Thus, we find that:

$$[m]\psi'(C) = (\psi'\hat{\psi})(\psi(C)) = [n]\psi(C)$$

for some nonzero integer n. Thus, we find that $\mathrm{Hom}(J(C), A)$ is a torsion-free, rank 1 module over $\mathbb{Z}$. Since it is finitely generated, it must be a free module of rank 1, so let $\Psi \colon J(C) \to A$ be a generator. Then for every rational map $\phi \colon C \to A$ induced by a homomorphism from $J(C)$ to A, there exists an integer n such that:

$$\phi(C) = [n]\Psi(\pi(C))$$

Thus, modulo the endomorphisms of A, there are only finitely many curves on A which are birational to C and lie in V.

It remains only to show that there are only finitely many curves C whose Jacobians $J(C)$ are isogenous to A. By Theorem 18.1 of [8], a given abelian variety admits only a finite number of isomorphism classes of principal polarisations, and by Corollary 12.2 of [9], a curve of genus two (or greater) is determined up to isomorphism by its Jacobian and principal polarisation. By a theorem of Faltings [5], we know that there are only finitely many isomorphism classes of abelian varieties which are k-isogenous to A. The result follows. ♣

Note that the same trick will not work for the curves of genus four or higher, since by Proposition (8) of [4], there are infinitely many hyperelliptic curves on A which are not related by translation or endomorphisms of A. It is not clear to the author whether the proof of Corollary 3.4 can be modified for the set of curves of genus three on A.

4 Theorems 3.2 and 3.3

In this section, we prove Theorems 3.2 and 3.3. Since the definitions of a sparse and stratified set do not depend on the choice of big height on X, we will take H to be the height on X induced by the Néron-Tate canonical height on A relative to some fixed even line bundle. Note that $H(P) \geq 1$ for all $P \in U(k)$, where $U \subset X$ is the image of the complement of the set of 2-torsion points of A.

Lemma 4.1 *Let S be any subset of $U(k)$, with counting function bounded by a constant multiple of a power of B:*

$$N_S(B) = \#\{P \in S \mid H(P) \leq B\} \leq cB^\delta$$

with $\delta > 0$. Then for any positive integer m, the image $\widetilde{[m]}S$ of S under the rational endomorphism $\widetilde{[m]}$ of U induced by multiplication-by-m satisfies:

$$N_{E(S)-T}(B) = \#\{P \in \widetilde{[m]}S \mid H(P) \leq B\} \leq c^{1/m^2} B^{\delta/m^2}$$

where T is the set of k-rational points of height 1 on X.

Proof: Assume that $H(\widetilde{[m]}P) \leq B$. Then we get:

$$H(P) \leq B^{1/m^2}$$

by the elementary properties of the multiplicative canonical height. By hypothesis, there are only $c^{1/m^2} B^{\delta/m^2}$ points which satisfy this hypothesis, as desired. ♣

Note that the set T is finite, since there are only finitely many points of $U(k)$ with bounded canonical height. We now prove the main technical theorem of the section, from which Theorems 3.2 and 3.3 will follow.

Theorem 4.2 *Let S be a subset of $X(k)$, and let $\delta > 0$ be a positive real number. Assume that $N_S(B) = O(B^\delta)$. Then $N_{E(S)}(B) = O(B^\delta)$. Furthermore, for any $\epsilon > 0$, there exists an integer $n \in \mathbb{Z}$ such that:*

$$N_{E(S)-S_n} = O(B^\epsilon)$$

where

$$S_n = \bigcup_{m=1}^{n} \widetilde{[m]}S$$

Proof: By hypothesis, there is a positive constant c such that for all $B \geq 1$:

$$N_S(B) \leq cB^\delta$$

We wish to show that $N_{E(S)} = O(B^\delta)$. By definition of X, we have $\widetilde{[m]} = \widetilde{[-m]}$. We therefore have the following relation on counting functions:

$$N_{E(S)}(B) \leq \sum_{m=1}^{\infty} N_{\widetilde{[m]}S}(B)$$

Of all the points of S whose height is greater than one, there must exist a point Q of minimal height – denote the height of Q by ℓ. Then for all $m > (\log B/\log \ell)^{1/2}$, the only points of $\widetilde{[m]}S$ of height less than B are points of height 1 – that is to say, points of the finite set T. Thus, we can replace the above infinite sum with a finite one:

$$N_{E(S)}(B) \leq \sum_{m=1}^{\alpha} N_{\widetilde{[m]S}}(B)$$

where $\alpha = \lfloor (\log B / \log \ell)^{1/2} \rfloor$. We can now calculate as follows:

$$\begin{aligned} N_{E(S)}(B) &\leq \sum_{m=1}^{\alpha} N_{\widetilde{[m]S}}(B) \\ &\leq \#T + \sum_{m=1}^{\alpha} (cB^{\delta})^{1/m^2} \\ &\leq \#T + cB^{\delta} + (\log B/\log \ell)^{1/2}(cB^{\delta})^{1/4} \\ &= O(B^{\delta}) \end{aligned}$$

as desired.

Furthermore, if $\epsilon > 0$, then for $n > \sqrt{\delta/\epsilon}$, we get:

$$N_{E(S)-S_n} \leq \sum_{m=n}^{\alpha} N_{\widetilde{[m]S}}(B) = O(B^{\epsilon})$$

which is precisely the second assertion in the theorem. ♣

Theorem 4.2 immediately implies Theorem 3.2. We proceed to deduce Theorem 3.3:

Proof of Theorem 3.3*:* Say S is a stratified set, and let $\epsilon > 0$ be any real number. Then we can find a finite union $C(\epsilon)$ of rational curves such that $N_{S-C(\epsilon)}(B) = O(B^{\epsilon})$. By Theorem 4.2, it follows that $N_{E(S-C(\epsilon))}(B) = O(B^{\epsilon})$ as well. Thus, it suffices to show that $E(C(\epsilon))$ is a stratified set. This follows immediately from the second assertion of Theorem 4.2. ♣

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