

A CONJECTURE ON RATIONAL APPROXIMATIONS TO RATIONAL POINTS

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ABSTRACT. In this paper, we examine how well a rational point P on an algebraic variety X can be approximated by other rational points. We conjecture that if P lies on a rational curve, then the best approximations to P on X can be chosen to lie along a rational curve. We prove this conjecture for a wide range of examples, and for a great many more examples we deduce our conjecture from Vojta's Main Conjecture.

1. INTRODUCTION

The distribution of rational points on an algebraic variety X is very subtle. For example, it can often happen that there is a proper subvariety Y of X such that the set of rational points of Y has density one in the set of rational points of X , where density is used in the sense of Weil heights. Roughly speaking, such subvarieties Y are called accumulating subvarieties. Thus, if one is interested in the arithmetic of X , one must first identify which rational points of X lie on Y , and which do not.

Unfortunately, even the purely geometric problem of identifying potential accumulating subvarieties Y can be very difficult. It would be helpful to have a local and arithmetic criterion to identify points P which lie on accumulating subvarieties. In this paper, we do not quite manage to construct such a criterion, but we do identify an invariant, called the approximation constant of P on X with respect to a divisor D , which describes how well P can be approximated by other rational points of X . (See Definition 2.5 for details.) If P lies on an accumulating subvariety, then it can be well approximated by other rational points, and therefore will have a small approximation constant.

In the course of computing this constant in many examples, it became clear that sequences which best approximate a rational point P tend to lie along curves. While there are counterexamples to show

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that this principle cannot hold in general, we are able to formulate a conjecture which predicts that it should be true whenever P lies on a rational curve defined over the field of coefficients (see Conjecture 2.7). We prove this conjecture for a wide range of examples.

The basic technique we use for proving Conjecture 2.7 is inductive. We start by proving the conjecture for \mathbb{P}^n (see Theorem 2.1). We then prove a number of inductive results which enable us to take advantage of the structure of the Néron-Severi group of X . Using these inductive results, we prove Conjecture 2.7 for a wide range of rational surfaces by means of a careful analysis of the nef cone.

In the final section, we make several remarks about further directions in which to attack Conjecture 2.7, and drawbacks of our current techniques. We describe some cases which are unlikely to be proven using our current techniques, and we further deduce Conjecture 2.7 from Vojta's Main Conjecture (Conjecture 3.4.3 of [Vo]) for a generic variety of non-negative Kodaira dimension.

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2. LINES

In this paper, all heights are absolute and multiplicative. We fix a number field k .

Theorem 2.1. *Let $P \in \mathbb{P}^n(k)$ be any k -rational point. Let $\{P_i\}$ be any sequence of k -rational points, and let S be the set of archimedean places of k , with the convention that pairs of complex conjugate embeddings count as one place. Suppose there is a positive real constant $c \in \mathbb{R}$ such that for all i , we have:*

$$\left(\sum_{v \in S} \text{dist}_v(P, P_i) \right) H(P_i) \leq c$$

where the function dist_v denotes the local distance function on $\mathbb{P}^n(k)$ induced by v . Then the set $\{P_i\}$ is a subset of the finite union of lines through P of Plücker height at most $N(P)c$, where $N(P)$ is a constant which does not depend on c .

Proof: Without loss of generality, we may assume that $P = [0 : 0 : \dots : 0 : 1]$, and that none of the P_i lie on the line $x_n = 0$. Fix a set R of representatives of the class group of k .

Write $P_i = [a_{i1} : \dots : a_{in} : b_i]$, where the a_{ij} and b_i are integers (that is, elements of \mathcal{O}_k) with $\gcd(a_{i1}, \dots, a_{in}, b_i) \in R$. Let $v \in S$ be any archimedean place of k . Up to multiplication by a bounded function (which won't affect the conclusion of the theorem), we may write the v -distance as:

$$\text{dist}_v(P, P_i) = \max_j \{|a_{ij}/b_i|_v\}$$

and the height as:

$$H(P_i) = \left(\prod_{v \in S} \max\{|a_{i1}|_v, \dots, |a_{in}|_v, |b_i|_v\} \right)^{1/[k:\mathbb{Q}]}$$

Note that with our choice of representation, the non-archimedean absolute values do not contribute more than a (multiplicatively) bounded function to the height.

Now suppose that $\sum_{v \in S} \text{dist}_v(P, P_i) H(P_i) \leq c$ for all i . Then we get:

$$\left(\sum_{v \in S} \max\{|a_{ij}/b_i|_v\} \right)^{[k:\mathbb{Q}]} \prod_{v \in S} \max\{|a_{i1}|_v, \dots, |a_{in}|_v, |b_i|_v\} \leq c^{[k:\mathbb{Q}]}$$

and hence

$$\left(\sum_{v \in S} \max\{|a_{ij}|_v\} \right)^{[k:\mathbb{Q}]} \max\{|a_{i1}/b_i|_v, \dots, |a_{in}/b_i|_v, 1\} \leq c^{[k:\mathbb{Q}]}$$

and so *a fortiori*, we obtain:

$$\sum_{v \in S} \max\{|a_{ij}|_v\} \leq c$$

In particular, there are only finitely many choices for each a_{ij} , and each choice corresponds to a line of height at most $c^{[k:\mathbb{Q}]}$. ♣

In light of this theorem, we make the following definitions:

Definition 2.2. Let $X \subset \mathbb{P}_k^n$ be an algebraic variety defined over k , and let S be the set of archimedean absolute values on k . Define:

$$\text{dist}(P, Q) = \sum_{v \in S} \text{dist}_v(P, Q)$$

for any rational points P and Q on X . Note that this is a well defined distance function on the metric space $\prod_{v \in S} X(k_v)$, where k_v is the completion of k with respect to v .

In this paper, all limits will be computed with respect to the topology induced by the distance function dist .

Definition 2.3. *Let X be an algebraic variety defined over k , and let $P \in X(k)$ be any rational point. Let D be any divisor on X , with corresponding height function H_D . Assume that there is a positive constant c such that $H_D(Q) > c$ for all Q in some Zariski open neighbourhood of P . (This will be satisfied if, for example, D is ample, or more generally, if some multiple of D is basepoint free.) For any sequence $\{P_i\} \subset X(k) - \{P\}$ with $P_i \rightarrow P$, define the approximation constant on X of $\{P_i\}$ with respect to P and D to be the smallest non-negative real number α such that:*

$$\limsup_{i \rightarrow \infty} \text{dist}(P, P_i)^\alpha H_D(P_i) < \infty$$

If there is no such smallest non-negative real number, then the sequence does not have an approximation constant.

Notice that although the definition of dist depends on the choice of embedding of X in projective space, the definition of the approximation constant does not. This is because two different embeddings of X are diffeomorphic, and hence distances change by no more than a multiplicative function bounded away from zero and infinity. In this paper, this ambiguity will never be significant, so we will refer to dist as a function independent of the embedding of X .

Example 2.4. For example, let $P = [0: 0: 1] \in \mathbb{P}^2$, and let $f(n) = n/\log n$. On any \mathbb{Q} -rational line L through P , we can find an infinite sequence of rational points $\{P_i\}$ such that $\text{dist}(P, P_i)H(P_i) \leq c_L$, where c_L is a constant that depends on L . (See for example Theorem 2.1 of [M1].) Therefore, there is a positive constant C and an infinite sequence $\{Q_n\} \subset \mathbb{P}^2$ of rational points such that $\text{dist}(Q_n, P)f(H(Q_n)) < C$ for all n . Moreover, we can choose the points $\{Q_n\}$ such that no two of them are collinear with P . Thus, by Theorem 2.1, the approximation constant of this sequence, if it exists, is greater than one. But it is clear that for every $\epsilon > 0$, there exists a positive constant C_ϵ such that $\text{dist}(Q_n, P)^{1+\epsilon}H(Q_n) < C_\epsilon$, so any constant of approximation for this sequence is at most one. We conclude that the sequence $\{Q_n\}$ does not have a constant of approximation.

Definition 2.5. *Let X , P , D , and H_D be as in Definition 2.3. Define the approximation constant of P on X with respect to D to be the minimum (if it is achieved) of all approximation constants on X of sequences $\{P_i\} \subset X(k) - \{P\}$ with respect to D . A sequence of best approximation to P with respect to D is a sequence whose corresponding*

approximation constant is equal to the approximation constant of P . A curve of best approximation to P is a curve on X passing through P that contains a sequence of best approximation to P .

Notice that by this definition, sequences with no approximation constant (such as the one described in Example 2.4) cannot be a sequence of best approximation to a point P . If Conjecture 2.7 is true, then Theorem 2.8 makes it clear that this is a reasonable restriction.

We may therefore improve Theorem 2.1 as follows.

Theorem 2.6. *Let $P \in \mathbb{P}^n(k)$ be any rational point. Then there exists a sequence of best approximation to P . Moreover, any such sequence is a subset of a finite set of lines and has constant of approximation equal to 1, and any line through P is a curve of best approximation to P .*

Proof: We first prove the theorem for \mathbb{P}^1 . Without loss of generality, we may assume that $P = [0 : 1]$. Fix a set of representatives R of the class group of k . Let $Q = [a : b]$ be any point in $\mathbb{P}^1(k)$, where a and b lie in \mathcal{O}_k with $(a, b) \in R$ and $a \neq 0$. Since $\{[1/i : 1]\}$ is clearly a sequence with constant of approximation equal to 1, it suffices to show that $\text{dist}(Q, P)H(Q) \geq C_k$, where C_k is a constant depending only on k .

Since the arithmetic mean is always greater than the geometric mean, for Q close enough to P we compute:

$$\begin{aligned} \text{dist}(P, Q) &= \sum_v |a/b|_v \geq [k : \mathbb{Q}] \prod_v |a/b|_v^{1/[k:\mathbb{Q}]} \\ &= [k : \mathbb{Q}] (N(a)/N(b))^{1/[k:\mathbb{Q}]} \end{aligned}$$

where all sums and products are over archimedean places v , and N denotes the norm $N_{k/\mathbb{Q}}$. We also compute:

$$H(Q) = \left(\prod_v \max\{|a|_v, |b|_v\} \right)^{1/[k:\mathbb{Q}]} \geq N(b)^{1/[k:\mathbb{Q}]}$$

where the product again ranges over archimedean places v . Note that since $(a, b) \in R$, if we change the representation of Q , then we will change the ideal (a, b) by multiplication by an element of k^* . Since no two elements of R represent the same ideal class, it follows that the representation changes by multiplication by a unit of \mathcal{O}_k , and so $H(Q)$ does not change. Since $a \in \mathcal{O}_k$ is nonzero, it follows that $N(a) \geq 1$, so that:

$$\text{dist}(P, Q)H(Q) \geq [k : \mathbb{Q}]$$

and we have proven the result for \mathbb{P}^1 .

In general, if $P \in \mathbb{P}^n$ is any rational point, we can find an embedding $\phi: \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ such that $\phi(\mathbb{P}^1)$ is a line, and $\phi(0 : 1) = P$. Since ϕ

only changes distances and heights by multiplication by a function bounded away from zero and infinity, it follows that a sequence of best approximation to P along the line $\phi(\mathbb{P}^1)$ has constant 1. But by Theorem 2.1, any sequence with approximation constant at most 1 must lie in a finite set of lines through P . The result follows. ♣

In light of Theorem 2.1, we make the following conjecture:

Conjecture 2.7. *Let V be an algebraic variety defined over k , and D any ample divisor on V . Let P be any k -rational point on V . Assume that there is some rational curve C on V , defined over k , such that $P \in C(k)$. Then a sequence of best approximation to P on V with respect to D exists, and may be chosen to lie along some rational curve through P .*

It is easy to see that if we remove the hypothesis that P lie on a rational curve, then the conjecture is false. Consider the case of a simple abelian variety V of dimension greater than one, and P any k -rational non-torsion point. Then P is certainly a limit point of the set $V(k)$ in the archimedean topology, but any curve through P must have geometric genus at least two, which by Faltings' Theorem can contain only finitely many k -rational points.

The following theorem is quite useful:

Theorem 2.8. *Let $C \subset \mathbb{P}^n$ be any irreducible rational curve of degree d defined over k , and let $P \in C(k)$ be any k -rational point. Let $f: \mathbb{P}^1 \rightarrow C$ be the normalization map, and let m be the maximum multiplicity of a branch of f over P . Then a sequence of best approximation to P on C has constant of approximation d/m .*

Proof: If P is a smooth point of C , then the result follows trivially from the observation that f changes distances to P only by a multiplicative function bounded away from zero and infinity, and f raises heights to the power d (up to a multiplicative function bounded away from zero and infinity).

If P is a singular point of C , then up to a multiplicative function bounded away from zero and infinity, we have $\text{dist}(f(P), f(Q)) = \text{dist}(R, Q)$ near P , where R is some point in the finite set $f^{-1}(P)$. Even though the point R depends on the point Q near P , it nevertheless provides only a finite number of alternatives for $\text{dist}(f(P), f(Q))$, so that any sequence of best approximation to P must have an infinite subsequence which is a sequence of best approximation to some element of $f^{-1}(P)$.

Thus, let $R \in f^{-1}(P)$ be any point, and let m be the multiplicity of the branch of f through R . Assume without loss of generality that

$R = 0$. Near R , in affine coordinates, the function f can be written as $f(x) = (f_1(x), \dots, f_n(x))$, where $C \subset \mathbb{P}^n$ and the f_i are rational functions in x . The multiplicity m_R of f at R is equal to $\min_i \{\text{ord}_0(f_i(x))\}$, where $\text{ord}_0(f_i)$ denotes the largest power of x which divides the numerator of f_i (in lowest terms). Let i be some index which achieves the minimum, and write $f_i(x) = x^{m_R} + O(x^{m_R+1})$. Then as $Q = x$ approaches $R = 0$, the distance between Q and R is $\sum_v |x|_v$, and the distance between $f(Q)$ and $f(R)$ is (up to a bounded multiplicative constant) $\sum_v |x^{m_R}|_v \ll (\sum_v |x|_v)^{m_R}$. The result now follows from Theorem 2.1. ♣

Theorem 2.8 explains why Conjecture 2.7 does not refer specifically to rational curves of minimal D -degree. A rational point P may be better approximated along a curve of higher degree on which it is a cusp of high multiplicity than along a smooth curve of much lower degree.

Example 2.9. Let C be the plane curve $y^2z = x^3 \subset \mathbb{P}^2$, and P the cusp $[0 : 0 : 1]$. For $k = \mathbb{Q}$, it is not hard to see that a sequence of best approximation to P along C is given by the sequence $[1/B^2 : 1/B^3 : 1]$, which has constant of approximation $3/2$.

However, let $f: \mathbb{P}^1 \rightarrow C$ be the normalization map. Then $f^*\mathcal{O}(1)$ is $D = \mathcal{O}(3)$. A sequence of best D -approximation to P along \mathbb{P}^1 has constant 3, not $3/2$, since D corresponds to an embedding of \mathbb{P}^1 in \mathbb{P}^3 as a twisted cubic curve. This is because in Definition 2.3, we define dist in terms of an embedding of \mathbb{P}^1 , not an arbitrary morphism, whereas in this example, we deal with the distance inherited from \mathbb{P}^2 .

In other words, if $\tilde{W} \rightarrow W$ is the normalization of a subvariety of a smooth variety V , then the function $\text{dist}: \tilde{W} \times \tilde{W} \rightarrow \mathbb{R}$ may not agree with the pullback of the function $\text{dist}: V \times V \rightarrow \mathbb{R}$ to $\tilde{W} \times \tilde{W}$. If W is a curve, then the proof of Theorem 2.8 describes how these two functions differ, but if W has higher dimension, then one must be more careful.

Despite the unpleasant possibility described by Example 2.9, we can relate Conjecture 2.7 to the problem of finding accumulating curves through a point P .

Definition 2.10. Let $S \subseteq \mathbb{P}^n(k)$ be any set of k -rational points. The counting function for S is defined to be:

$$N_S(B) = \#\{P \in S \mid H(P) \leq B\}$$

where H denotes the standard height function on \mathbb{P}^n .

Definition 2.11. Let $V \subseteq \mathbb{P}^n$ be an algebraic variety defined over k , and let $W \subset V$ be any proper closed subset. Then W is said to be an

accumulating subvariety of V if and only if:

$$N_{V-W}(B) = o(N_W(B))$$

where by N_{V-W} and N_W we mean $N_{V(k)-W(k)}$ and $N_{W(k)}$, respectively. Roughly speaking, an accumulating subvariety of V is a proper closed subset W such that asymptotically, there are more rational points of bounded height on W than there are off W .

Theorem 2.12. *Let $V \subseteq \mathbb{P}^n$ be a smooth algebraic variety defined over k , and let $P \in V(k)$ be any k -rational point. Assume that Conjecture 2.7 is true for P on V — that is, assume that there is a rational curve C through P which contains a sequence of best approximation to P . Assume further that C has only ordinary singularities at P . If P lies on an accumulating curve of V , then that curve contains C as an irreducible component.*

Proof: First, note that since V contains a rational curve C through P , it follows that any accumulating curve containing P must be a rational curve, and moreover any component containing P must be a rational curve of minimal degree through P . By Theorem 2.8, since C has only ordinary singularities through P , it is clear that any rational curve through P must have degree at least $\deg C$. Thus, C is a rational curve of minimal degree through P , and therefore any accumulating curve through P must contain C as an irreducible component. ♣

3. OTHER VARIETIES

Next, we consider the question of rational approximation of rational points on more general varieties. We begin with a straightforward but surprisingly useful result on products of varieties:

Theorem 3.1. *Let X and Y be smooth algebraic varieties defined over a number field k , and let P and Q be k -rational points on X and Y , respectively. Let L_X and L_Y be divisors on X and Y , respectively, whose corresponding height functions H_X and H_Y are bounded below by a positive constant in some Zariski open neighbourhoods of P and Q , respectively. Let $\{P_i\}$ (respectively $\{Q_i\}$) be a sequence of best L_X -approximation (respectively best L_Y -approximation) for P (respectively Q), with constant of approximation α (respectively β). Then either $\{(P_i, Q)\}$ or $\{(P, Q_i)\}$ is a sequence of best L -approximation for (P, Q) on $X \times Y$, where $L = \pi_1^*L_X \otimes \pi_2^*L_Y$ (π_i is the projection onto the i th factor). Moreover, any sequence $\{(S_i, T_i)\}$ whose constant of approximation is less than $\alpha + \beta$ must have either $S_i = P$ or $T_i = Q$ for all but finitely many i .*

Proof: For any k -rational point (S, T) on $X \times Y$, we have $H_L(S, T) = H_X(S)H_Y(T)$, where H_X and H_Y are heights on X and Y with respect to the divisors L_X and L_Y , respectively. To prove the theorem, note that the non-negativity of α and β implies that it suffices to prove the last claim.

Let $\{(S_i, T_i)\}$ be any sequence of points in $(X \times Y)(k) - \{(P, Q)\}$ with $(S_i, T_i) \rightarrow (P, Q)$. We know that if $\{S_i\}$ is infinite then $\text{dist}(P, S_i)^{\alpha-\epsilon} H_X(S_i)$ is unbounded for any $\epsilon > 0$. Similarly, if $\{T_i\}$ is infinite then $\text{dist}(Q, T_i)^{\beta-\epsilon} H_Y(T_i)$ is unbounded for any $\epsilon > 0$. Thus, if there are infinitely many i for which both $S_i \neq P$ and $T_i \neq Q$, then we deduce that the following function is unbounded:

$$\max\{\text{dist}(P, S_i), \text{dist}(Q, T_i)\}^{\alpha+\beta-\epsilon} H_X(S_i) H_Y(T_i)$$

We may now deduce that the following function is unbounded:

$$\text{dist}((P, Q), (S_i, T_i))^{\alpha+\beta-\epsilon} H_L(S_i, T_i)$$

which is to say that $\{(S_i, T_i)\}$ has constant of approximation at least $\alpha + \beta$. ♣

We have the following immediate corollary:

Corollary 3.2. *Let X be a variety defined over k , and let $P \in X(k)$ be any k -rational point. Let D_1 and D_2 be two divisors on V with height functions H_1 and H_2 bounded below by a positive constant in a neighbourhood of $P \in X(k)$. Assume that C_i is a curve of best D_i -approximation to P for each i , and let $D = a_1 D_1 + a_2 D_2$ be any positive linear combination of D_1 and D_2 .*

- *If $C_1 = C_2 = C$, then C is also a curve of best D -approximation to P .*
- *If $C_1.D_2 = 0$, then either C_1 is a curve of best D -approximation to P , or $C_2.D_1 = 0$ and C_2 is a curve of best D -approximation to P .*

The following theorem will be used in the proofs of Theorem 3.17 and Theorem 3.9.

Theorem 3.3. *Let X be a variety defined over k , and let D and E be divisors on X . Let $P \in X(k)$ be a rational point, and C a curve of best D -approximation to P . Assume that H_D and H_{D+E} are bounded below by positive constants in some neighbourhood of P , that E is effective, and that $C \cap E = \emptyset$. Then C contains a sequence of best $(D + E)$ -approximation to P .*

Proof: Since $C \cap E = \emptyset$, we can find a real, positive constant α such that $H_E(Q) \in [1/\alpha, \alpha]$ for all $Q \in C(k)$. Thus, the sequence of

best D -approximation to P along C will have the same constant of approximation with respect to $D + E$ as with respect to D . However, $\log H_E$ is bounded from below away from E , and $P \notin E(k)$. It follows that for any sequence $\{Q_i\}$ of rational points on V with $Q_i \rightarrow P$, if the quantity $\text{dist}(P, Q_i)^\gamma H_D(Q_i)$ is unbounded, then the quantity $\text{dist}(P, Q_i)^\gamma H_{D+E}(Q_i)$ will also be unbounded. The theorem follows. ♣

These theorems, surprisingly, give an immediate proof of Conjecture 2.7 for any split, geometrically minimal surface X over k . We say that a surface X is split over k if and only if the inclusion of Néron-Severi groups $NS(X_k) \rightarrow NS(X_{\mathbb{C}})$ is an isomorphism – that is, if every algebraic cycle on $X_{\mathbb{C}}$ is numerically equivalent to some k -rational cycle. The proof relies crucially on the classification of minimal rational surfaces over \mathbb{C} , which can be found in, for example, [Be].

Corollary 3.4. *Conjecture 2.7 holds for any split, geometrically minimal rational surface defined over k . In particular, it holds for all the Hirzebruch surfaces H_n for $n \geq 0$.*

Proof: The classification shows that any minimal rational surface over \mathbb{C} is either \mathbb{P}^2 , $H_0 = \mathbb{P}^1 \times \mathbb{P}^1$, or a Hirzebruch surface H_n for some integer $n > 0$. The case of \mathbb{P}^2 is proven in Theorem 2.1. The case $\mathbb{P}^1 \times \mathbb{P}^1$ is immediate from Theorem 3.1. The case of H_n follows from Theorem 3.1 and Corollary 3.2 as follows. Let $A(H_n)$ be the closure in $N(H_n) = NS(H_n) \otimes \mathbb{R}$ of the cone of ample divisors on H_n , where $NS(H_n)$ is the Néron-Severi group of H_n . Then $A(H_n)$ is generated by the two divisor classes $D = S + nF$ and F , where $F = \pi^*(P)$ is a fibre of the map $\pi: H_n \rightarrow \mathbb{P}^1$ and S is the unique section of π with self-intersection $-n$. The class D is $f^*(L)$, where f is the contraction of S to the vertex of the cone C over a rational normal curve of degree n and L is a line of the ruling of the cone.

Consider the classes D , F , and $D + F$. For any rational point P on H_n , it is clear that a curve of best F -approximation to P is a fibre of π , and a curve of best D -approximation to P is the preimage of a line through $f(P)$ on the cone C (if $P \notin S(k)$) or S (if $P \in S(k)$). If $P \notin S(k)$, then these are the same curve, so we are done by Corollary 3.2. If $P \in S(k)$, then we simply note that the divisor $D + F$ corresponds to an embedding of H_n in projective space such that fibres of π are all lines, and S is also a line, so they both contain sequences of best $(D + F)$ -approximation to P . We again conclude by Corollary 3.2, since every very ample divisor on H_n is a positive linear combination either of D and $D + F$ or F and $D + F$. ♣

Theorem 3.5. *Let $n \geq 2$, and let X be the blowup of the Hirzebruch surface H_n at $k < n$ points, no two of which lie in the same fibre of the map to \mathbb{P}^1 . Conjecture 2.7 is true for X .*

Proof: First, it clearly suffices to assume that none of the blown up points lies on the $(-n)$ -section on H_n , since blowing up points on this section will only increase n .

Let S be the class of the strict transform of the $(-n)$ -section on H_n , and let F be the class of the total transform of a fibre of the map from H_n to \mathbb{P}^1 . For each of the k reducible fibres, let E_i and F_i be the two components, of which F_i is the one which intersects S .

Let $\alpha \in \{0, 1\}^k$ be any vector, and define a divisor D_α on X by:

$$D_\alpha = S + nF - \alpha \cdot (E_1, \dots, E_k)$$

where the \cdot denotes a formal dot product.

Claim 3.6. *The effective cone of X is generated by the divisors E_i , F_i , and S . The nef cone of X is generated by the divisors D_α and F , where α ranges over all of $\{0, 1\}^k$.*

Proof: The Picard rank of X is $k + 2$, so an arbitrary divisor on X can be written as $D = aS + bF + \sum f_i E_i$. If D is to be ample, then its intersection with S , E_i , and F_i must be positive, giving:

$$b - na > 0 \qquad a + f_i > 0 \qquad f_i < 0$$

These inequalities define an open cone in \mathbb{R}^{k+2} – let C be the closure of this cone. Then C is finitely generated, and the generating rays of C are all intersections of $k + 1$ of the hyperplanes $V = \{b - na = 0\}$, $V_i = \{a + f_i = 0\}$, and $W_i = \{f_i = 0\}$.

Let \mathbf{v} be a generator of C . If $\mathbf{v} \notin V$, then for some j , $\mathbf{v} \in V_j \cap W_j$, since \mathbf{v} must be contained in at least $k + 1$ of the listed hyperplanes. Thus, \mathbf{v} is contained in the hyperplane $a = 0$. Furthermore, since \mathbf{v} is an extremal ray of C , there must be some set of exactly $k + 1$ of the V_i and W_i whose intersection is the space generated by \mathbf{v} . Since the vector F is contained in the intersection of all the V_i and W_i , it follows that \mathbf{v} is a positive multiple of F .

Thus, we may assume that $\mathbf{v} \in V$. If $\mathbf{v} \in V_j \cap W_j$ for some j , then \mathbf{v} is again contained in $a = 0$, and thus also in $b = 0$. But there must also be some ℓ such that $\mathbf{v} \notin V_\ell \cup W_\ell$, and hence $f_\ell > 0$ and $f_\ell < 0$. This is clearly a contradiction, and so for all i , $\mathbf{v} \notin V_i \cap W_i$.

Let $\alpha \in \{0, 1\}^k$ be the vector whose i th component is 0 if $\mathbf{v} \in V_i$, and 1 if $\mathbf{v} \in W_i$. Then $D_\alpha \in V_i$ if and only if $\mathbf{v} \in V_i$, and similarly for W_i . Since $D_\alpha \in V$, the independence of the $k + 1$ hyperplanes shows that \mathbf{v} is a positive multiple of D_α .

Thus, the two cones described in the claim are indeed dual to one another. To conclude the proof of the claim, it suffices to show that every positive linear combination of the D_α and F is ample. First, it is clear that F and D_α are basepoint free, since F corresponds to a morphism to \mathbb{P}^1 , and D_α corresponds to the morphism from X to a cone, blowing down S and exactly one component of each reducible fibre. We next note that $F^2 = 0$ and $F.D_\alpha = 1$, and:

$$\begin{aligned} D_\alpha.D_\beta &= (S + nF)^2 + (\alpha \cdot (E_1, \dots, E_k))(\beta \cdot (E_1, \dots, E_k)) \\ &= n - (\alpha \cdot \beta) \\ &\geq n - k \end{aligned}$$

which is positive. Therefore, by the Nakai-Moishezon criterion ([Ha], Theorem V.1.10), any positive linear combination of F and the D_α is ample, and thus the claim is proven. ♣

We now prove Conjecture 2.7 for X . If P does not lie on S or any reducible fibre, then for each α , a sequence of best D_α -approximation to P is clearly contained in the component C of F , and certainly a sequence of best F -approximation is also contained in C . Thus, for any ample D , C is a curve of best D -approximation to P .

If P lies on exactly one E_i or F_i (and not S), then the same analysis shows that this same E_i or F_i is a curve of best D -approximation to P for any ample D . If P lies on S but no E_i or F_i , then notice that $S.D_\alpha = 0$ for any α , so we conclude that S is a curve of best D_α -approximation to P for all α . Since $S.F = 1$, and since $D_\alpha + F$ contracts no curves through P , we conclude from Theorem 2.6 that S and C are both curves of best $(D_\alpha + F)$ -approximation to P , for any α . Since any ample divisor S is either a positive linear combination of elements of the set $\{D_\alpha, F + D_\alpha\}$, or a positive linear combination of elements of the set $\{F, F + D_\alpha\}$, we conclude from Corollary 3.2 that either S or C is a curve of best D -approximation to P for any ample D .

If $P = S \cap E_i$ for some i , then a similar analysis shows that for any ample divisor D , either S or E_i is a curve of best D -approximation to P . In particular, if we divide the nef cone of X into two subcones according to which of S or E_i has smaller degree, then it is straightforward to check that the curve of smaller degree has degree zero or one with respect to each generator of the corresponding subcone, and so by Theorem 2.6 and Corollary 3.2, we conclude that the curve of smaller D -degree is always a curve of best D -approximation to P . If $P = E_i \cap F_i$ for some i , a similar calculation shows that either E_i or F_i — whichever

has smaller D -degree — is always a curve of best D -approximation to P . ♣

Theorem 3.5 can be used to show that any k -split rational surface of Picard rank at most three satisfies Conjecture 2.7.

Theorem 3.7. *Let X be a k -split rational surface, of Picard rank at most three. Then Conjecture 2.7 is true for X .*

Proof: If the Picard rank is one, then X is isomorphic to \mathbb{P}^2 . If the Picard rank is two, then either it's geometrically minimal, or else it's \mathbb{P}^2 blown up at a point P . In this case, however, $X = H_1$, and the result follows from Corollary 3.4.

If the Picard rank of X is three, then it must be the blowup of some H_n at some point P , with $n \geq 0$. If $n = 0$, then since any blowup of $H_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is also a blowup of H_1 , we may instead take $n = 1$. If $n > 1$, then we can apply Theorem 3.5 directly. Thus, we assume that $n = 1$, and therefore that X is the blowup of \mathbb{P}^2 at two different points.

We begin by finding generators for the effective cone of X . Let $\pi: X \rightarrow \mathbb{P}^2$ be the blowing down map, and let E_1 and E_2 be the two exceptional divisors. Let S be the strict transform of the line in \mathbb{P}^2 which joins the two blown up points. If we write $L = S + E_1 + E_2$, then L is the class of $\pi^*\mathcal{O}(1)$. Then $L - E_1$ and $L - E_2$ correspond to morphisms ψ_1 and ψ_2 to \mathbb{P}^1 . We have the following well known description of the effective and nef cones of X .

Claim 3.8. *The curves S , E_1 , and E_2 generate the closed cone $\overline{NE}(X)$ of effective divisors on X , and the classes L , $L - E_1$, and $L - E_2$ generate the nef cone of X .*

We now prove Conjecture 2.7 for X . Let $P \in X(k)$ be any point, and let D be any ample divisor. If $P \neq S \cap E_i$, let C_i be the unique irreducible component of the fibre of $\psi_i: X \rightarrow \mathbb{P}^1$ through P . Each of L , $L - E_i$, and $A = (L - E_1) + (L - E_2)$ either contracts C_i , or else contracts no curves through P and maps C_i to a line. (Note that if P lies on S , then $C_i = S$.) Thus, by Theorem 2.6 and Corollary 3.2, we conclude that C_i is a curve of best D -approximation to P for any D in the cone generated by L , $L - E_i$, and A .

This covers all cases except $P = S \cap E_i$ and D is a positive linear combination of L , $L - E_i$, and A . In this case, notice that by Theorem 3.1, both S and E_i are curves of best approximation to P with respect to $2L - E_i$ and $L + A$. Since L contracts E_i , this means that if D lies in the cone generated by L , $2L - E_i$, and $L + A$, then E_i is a curve of best D -approximation to P by Corollary 3.2. And since $L - E_i$

and A both contract S , we conclude again by Corollary 3.2 that S is a curve of best D -approximation to P if D is a positive linear combination of $L - E_i$, $2L - E_i$, A , and $L + A$. This concludes the proof of Conjecture 2.7 for X . ♣

Proceeding to the other extreme from Theorem 3.5, we now prove a theorem about the case in which the map to \mathbb{P}^1 has exactly one reducible fibre.

Theorem 3.9. *Let X be a smooth rational surface obtained by a succession of blowups of the Hirzebruch surface H_n , and let $\pi: X \rightarrow \mathbb{P}^1$ be the associated map. Assume that π has only one reducible fibre, with m components, and assume $m < n$. If every multiple component of the reducible fibre intersects at least two other components then Conjecture 2.7 is true for every point of X .*

Proof: If any of the blown-up points of H_n lie on the $(-n)$ -section S , then the strict transform of S will have strictly smaller self-intersection than $-n$, and X can be obtained as a blowup of H_r for some $r > n$. Thus, if we choose n large enough, we may therefore assume that none of the blown-up points of H_n lie on S .

Let us establish some notation:

- The components of the reducible fibre of π are denoted by E_1, \dots, E_m , and we write $E_0 = S$ for the unique $(-n)$ -section of π .
- The classes E_0, \dots, E_m are a basis for the Néron-Severi group $NS(X)$, and we denote by D_0, \dots, D_m the dual basis with respect to the intersection pairing. That is, $D_i \cdot E_j = \delta_{ij}$.
- We write $F = m_1 E_1 + \dots + m_m E_m$, so that m_i is the multiplicity of E_i as a component of the reducible fibre of π .

We proceed with a technical lemma.

Lemma 3.10. *Assume without loss of generality that $E_m^2 = -1$ and $E_m \cdot S = 0$, and let $f: X \rightarrow Y$ be the map which blows down E_m . Let $\phi: Y \rightarrow \mathbb{P}^1$ be the map satisfying $\pi = \phi \circ f$, let E'_0 be the $(-n)$ -section of ϕ , and let E'_1, \dots, E'_{m-1} be the components of the unique reducible section of ϕ , ordered so that $E'_i = f_*(E_i)$.*

Let $\{D'_0, \dots, D'_{m-1}\}$ be the dual basis to $\{E'_0, E'_1, \dots, E'_{m-1}\}$. Then $D_i = f^(D'_i)$ if $0 \leq i < m$.*

Furthermore, E_m can intersect either one or two other components E_i . If E_m intersects one component E_i , then $D_m = f^(D'_i) - E_m$. If E_m intersects E_i and E_j , then $D_m = f^*(D'_i) + f^*(D'_j) - E_m$.*

Proof of Lemma: The fact that E_m can intersect at most two components E_i follows from the fact that no point of Y can lie on more than two components of the reducible fibre. The D_i are defined by the property that $D_i.E_j = \delta_{ij}$, where we define $E_0 = S$. It is a straightforward matter to verify this equality for all i and j . ♣

The next step is to compute the effective cone of X .

Lemma 3.11. *Assume that for all i , some multiple of the divisor D_i is basepoint free. Then the effective cone of X is generated by the components $\{E_1, \dots, E_m\}$ of the reducible fibre and the $(-n)$ -section S of π . Furthermore, we have $D_0 = F$ and $D_1 = S + nF$.*

Proof of Lemma: Let E_1, \dots, E_m be the set of components of the reducible fibre, where E_1 is the unique component which intersects S . Let $A = (a_{ij})_{i,j=0}^m$ be the intersection matrix of the S and E_i ; that is, let $a_{ij} = E_i.E_j$, where $E_0 = S$. The only entry of A which is not 0 or ± 1 is $a_{00} = -n$. We therefore may regard A as a specialization of the matrix A_N with entries in $\mathbb{Z}[N]$, where $(A_N)_{ij} = a_{ij}$ except that $(A_N)_{00} = -N$. Specializing $N = n$ reduces A_N to A .

To check that $\{S, E_i\}$ generates the effective cone, it suffices to check that the dual cone is the nef cone. Since $\{S, E_i\}$ form a basis for the vector space $NS(X) \otimes \mathbb{R}$, the dual cone will be generated by the vectors $D_j = \sum b_{ij}E_i$, where $B_N = (b_{ij}(N))_{i,j=0}^m$ is the inverse of the matrix A_N , $b_{ij} = b_{ij}(n)$, and $E_0 = S$. To show that this dual is the ample cone, it suffices to show that every positive linear combination of the D_j is ample. By the Nakai-Moishezon criterion, since we assume that some multiple of each D_i is basepoint free, it suffices to show that $b_{ij} = D_i.D_j \geq 0$ for all i and j , and that for each i , there is some j for which $D_i.D_j > 0$.

First, note that $D_0 = F$, the class of a fibre of π . To see this, note that $D_0.E_i = 0$ for all $i > 0$, and therefore $D_0 = \lambda F$ for some λ by the non-degeneracy of the intersection pairing. Since $D_0.S = 1$, we conclude that $D_0 = F$. In particular, the first row and column of B are independent of N , and $b_{0j} > 0$ for all $j > 0$.

Now let $i > 0$. Since the only entry of A_N depending on N is $(A_N)_{00} = -N$, Cramer's Rule implies that the i th row \mathbf{r}_i of A_N is $\mathbf{r}_i = N\mathbf{r}'_i + \mathbf{r}_i^\circ$, where \mathbf{r}'_i and \mathbf{r}_i° are independent of N . Specializing to $N = n$, we obtain $D_i = nD'_i + D_i^\circ$. (Note that $F.E_j = 0$ for all $j > 0$, so that the $(0,0)$ -cofactor matrix of A_N has nontrivial kernel, and therefore zero determinant. We therefore conclude that the determinant of A_N is ± 1 , and in particular independent of N .) Furthermore, we know that if \mathbf{r}_j^* denotes that j th column of A_N^{-1} , then $\mathbf{r}_i \cdot \mathbf{r}_j^* = \pm D_i.E_j = \delta_{ij}$

is independent of N for all j , and hence $D'_i.E_j = 0$ for all $j > 0$. Thus, $D'_i = \lambda_i F$ for some λ_i .

Clearly $D_1 = S + nF$, since $(S + nF).S = 0$, $(S + nF).E_j = 0$ if $j > 1$, and $(S + nF).E_1 = 1$. Thus, $\lambda_1 = 1 > 0$. Furthermore, since $S + nF = D_1 = \sum_j b_{j1} E_j$, this means that $b_{1j} = b_{j1} = \lambda_j n$ is just n times the multiplicity of E_j in the fibre F — in particular, λ_j is a positive integer. It therefore remains only to show that the coefficient of E_j in D_i° is at least $-m\lambda_j$, and that at least one coefficient is strictly greater than $-m\lambda_j$. (Recall that m is the number of components in the reducible fibre.)

We proceed by induction on m . If $m \leq 1$, the result follows immediately from the preceding calculations, since D_0 and D_1 will be the full list of D_i 's. For general m , note that there must be some E_i with $E_i^2 = -1$ and $i \neq 1$ — without loss of generality, we may assume that $i = m$. Let $f: X \rightarrow Y$ be the map that blows down E_m . Then Y is a smooth rational surface which admits a map $\phi: Y \rightarrow \mathbb{P}^1$ such that $\phi \circ f = \pi$, and the unique singular fibre of ϕ has $m - 1$ components.

Let $E'_i = f_*(E_i)$ for $i < m$, and let $S' = f_*(S)$. Denote the dual basis of $\{S', E'_i\}$ by $\{D'_i\}$. By induction, the coefficient of E'_j in D'_i is at least $-(m - 1)\lambda_j$, since the multiplicity of E'_j in $f_*(F)$ is also λ_j . If $i < m$, then by Lemma 3.10, the coefficient of E_j in D_i° is also at least $-(m - 1)\lambda_j \geq -m\lambda_j$. If $i = m$, then the coefficient of E_j in D_i is at least $-(m - 1)\lambda_j - 1 \geq -m\lambda_j$.

Thus, if $m \leq n$, then $D_i.D_j \geq 0$ for all i and j . Since the pairing is non-degenerate, for each i there is some j for which $D_i.D_j > 0$. This concludes the proof of the lemma. ♣

Remark 3.12. Note that the proof of Lemma 3.11 also shows that $D_i.D_j > 0$ for all i and j except $i = j = 0$.

The next step is to prove that for each i , some multiple of D_i is basepoint free. Together with Lemma 3.11, this will complete the computation of the effective cone of X .

Define a graph $G = (V, E)$ with vertex set $V = \{E_1, \dots, E_m\}$ and edge set $E = \{(E_i, E_j) \mid E_i.E_j > 0\}$. Since $E_i^2 < 0$, this defines a simple graph, and moreover, it is a tree. Root this tree at E_1 ; that is, define a partial order on V by $E_j \succeq E_i$ if and only if there is a simple path (that is, a path with no repeated vertices) from E_1 to E_j which contains the vertex E_i . Call E_i a leaf of G if and only if $E_j \succeq E_i$ implies $i = j$. The hypothesis of the theorem is equivalent to demanding that the leaves of G have multiplicity one in the fibre F .

Lemma 3.13. *Let E_i be a leaf of G . Then D_i is basepoint free.*

Proof: We proceed by double induction on n and m . If $n = 1$ there is nothing to prove. If $m = 1$, then X is the Hirzebruch surface H_n and the lemma is clear. If $n = 2$, then we must have $m \leq 1$, and the result is again clear. Note also that if $m > 1$, then the sum of the multiplicities of the E_i with $E_i^2 = -1$ is always at least two, and that this property will be preserved by blowing up.

Now consider a general $n > 2$, and let E_i be a leaf of G . Since E_i has multiplicity one by hypothesis, there must be some other (-1) -curve E_j . (It is possible that $j = 1$.) Let $f: X \rightarrow Y$ be the map that blows down E_j . Then Y will also be a rational surface with a map $\phi: Y \rightarrow \mathbb{P}^1$ with at most one reducible fibre. The unique $(-\ell)$ -section of ϕ will satisfy $\ell \leq n$, and the reducible fibre will have $m - 1$ components. Therefore, by induction, the dual divisor D'_i to $f_*(E_i)$ is basepoint free. But $D_i = f^*(D'_i)$, so D_i is basepoint free as well. ♣

Lemma 3.14. *Assume that $E_i \succeq E_j$. Then $m_i D_j - m_j D_i$ is an effective divisor supported on the reducible fibre of π . Furthermore, if E_i and E_j intersect, then $m_i D_j - m_j D_i = \sum_{E_t \succeq E_i} m_t E_t$.*

Proof: Since $F.(m_i D_j - m_j D_i) = 0$, it follows that $m_i D_j - m_j D_i$ is supported on the reducible fibre of π . Thus, it remains only to show that it is effective. It suffices to prove the result in the case that $E_i.E_j = 1$; a simple induction will prove the general case from there. Furthermore, if we blow down a (-1) -curve E_ℓ on X to obtain Y , then by Lemma 3.10, the truth of the lemma for E_i and E_j on X will be equivalent to the truth of the lemma for the images of E_i and E_j on Y , unless i or j equals ℓ . Thus, by induction on the number m of components of the reducible fibre, we may assume that E_i or E_j is a (-1) -curve on X , that any (-1) -curve in the reducible fibre is either E_i or E_j , and that $i \neq 1$ (because $E_i \preceq E_j$ and $i \neq j$).

Say E_i is a (-1) -curve of multiplicity one. Then it must be a leaf of G , so E_j must be the unique other component which intersects E_i . In that case, E_j must also have multiplicity one, so let Y be the surface obtained by blowing down E_i . By applying Lemma 3.10, we see that $D_j - D_i = E_i$, and the lemma follows.

Now assume that E_i is a (-1) -curve of multiplicity $m_i > 1$. Then by hypothesis E_i cannot be a leaf, so it must intersect two curves E_j and E_ℓ , with $m_i = m_j + m_\ell$. We have $E_\ell \succeq E_i \succeq E_j$, so let $f: X \rightarrow Y$ be the map that blows down E_i . Then Y is also a rational surface with $(-n)$ -section and a single reducible fibre with $m - 1$ components, so $\{S', E'_1, \dots, E'_{m-1}\}$ are generators of the effective cone of Y , with $S' = f_*(S)$ and $E'_r = f_*(E_r)$ for any r . Let $\{D'_0, \dots, D'_{m-1}\}$ be the

dual basis of $\{S', E'_1, \dots, E'_{m-1}\}$ in $\text{NS}(Y)$. By Lemma 3.10, we have $D_i = D_j + D_\ell - E_i$. By induction on m , we have:

$$m_\ell D'_j - m_j D'_\ell = \sum_{E_t \succeq E_\ell} m_t E'_t$$

since the multiplicity of E_r is the same as the multiplicity of E'_r , for any $r \neq i$. By Lemma 3.10 again, pulling back to X via f gives:

$$m_\ell D_j - m_j D_\ell = \left(\sum_{E_t \succeq E_\ell} m_t E_t \right) + m_\ell E_i$$

and therefore:

$$\begin{aligned} m_i D_j - m_j D_i &= m_i D_j - m_j D_j - m_j D_\ell + m_j E_i \\ &= m_\ell D_j - m_j D_\ell + m_j E_i \\ &= \left(\sum_{E_t \succeq E_\ell} m_t E_t \right) + m_i E_i \\ &= \sum_{E_t \succeq E_i} m_t E_t \end{aligned}$$

as desired.

Finally, we treat the case that E_j is the only (-1) -curve in the reducible fibre. It is not a leaf, since $E_i \succeq E_j$. If $j = 1$, then since $E_1 = E_j$ has multiplicity one, then since $m > 1$ ($i \neq j$), there must be at least one other (-1) -curve E_t on X . Therefore $j \neq 1$, and let E_i and E_ℓ be the two components which intersect E_j . (Note that $j \neq 1$ because $E_i \preceq E_j$.) We have $E_i \succeq E_j \succeq E_\ell$, $D_j = D_i + D_\ell - E_j$, and $m_j = m_i + m_\ell$. By induction and Lemma 3.10, we compute:

$$m_i D_\ell - m_\ell D_i = \left(\sum_{E_t \succeq E_i} m_t E_t \right) + m_i E_j$$

and therefore:

$$\begin{aligned} m_i D_j - m_j D_i &= m_i D_i + m_i D_\ell - m_i E_j - m_j D_i \\ &= (m_i D_\ell - m_\ell D_i) - m_i E_j \\ &= \sum_{E_t \succeq E_i} m_t E_t \end{aligned}$$

and the lemma is proven. ♣

Lemma 3.15. *For any i , some positive multiple of D_i is basepoint free.*

Proof of lemma: We have already proven this in the case that $i = 0$ (proof of Lemma 3.11) or E_i is a leaf (see Lemma 3.13). By Lemma 3.14, we know that some multiple of each D_i is effective, and in particular,

the base locus of D_i is supported on the set $\cup_{E_t \succ E_i} E_t$. It therefore suffices to find an effective divisor E , linearly equivalent to some multiple of D_i , supported on the set $S \cup (\cup_{E_t \not\succeq E_i} E_t)$. We claim that for all i , there is an effective divisor C_i and a positive rational number α_i such that $D_i = (m_i D_1 - \alpha_i F) + C_i$, C_i is supported on the set $\cup_{E_t \not\succeq E_i} E_t$, and the base locus of a suitable multiple of $m_i D_1 - \alpha_i F$ is a subset of S . In particular, we will show that $\alpha_i/m_i \leq \ell_i$, where ℓ_i denotes the number of edges in the shortest path in the graph G from E_1 to E_i .

If $i = 1$, this is trivial. We proceed by induction on ℓ_i , the number of edges in the shortest path from E_1 to E_i in the rooted tree. Let E_j be the parent of E_i — that is, let E_j be the unique component with $E_j \prec E_i$ and $E_j \cdot E_i = 1$. Then by induction we can write $D_j = (m_j D_1 - \alpha_j F) + C_j$, where C_j is effective and supported on the set $\cup_{E_t \not\succeq E_j} E_t$, and the base locus of a suitable multiple of $m_j D_1 - \alpha_j F$ is a subset of S . By Lemma 3.14, we can write:

$$m_i D_j - m_j D_i = \sum_{E_t \succeq E_i} m_t E_t$$

We therefore deduce:

$$\begin{aligned} D_i &= (1/m_j)(m_i D_j - \sum_{E_t \succeq E_i} m_t E_t) \\ &= (1/m_j)(m_i m_j D_1 - m_i \alpha_j F + m_i C_j - \sum_{E_t \succeq E_i} m_t E_t) \\ &= m_i D_1 - (m_i \alpha_j / m_j + 1)F + m_i C_j + \sum_{E_t \not\succeq E_i} m_t E_t \end{aligned}$$

Set $\alpha_i = m_i \alpha_j / m_j + 1$. Since $\alpha_j / m_j \leq \ell_j$, it follows that $\alpha_i / m_i = \alpha_j / m_j + 1 / m_i \leq \ell_i = \ell_j + 1$. We have $\ell_i \leq m < n$ for all i , so a suitable multiple of $D_1 - \alpha_i / m_i F = S + (n - \alpha_i / m_i)F$ will be basepoint free away from S , as desired. This means that for each i , the base locus of a suitable multiple of D_i is contained in the union of S and $\cup_{E_t \not\succeq E_i} E_t$. But we have already proven that the base locus of a suitable multiple of D_i is contained in the union $\cup_{E_t \succ E_i} E_t$. Since these two sets are disjoint, we conclude that a large enough multiple of D_i is basepoint free, as desired. ♣

We now conclude the proof of Theorem 3.9. Choose any $P \in X(k)$.

Claim 3.16. *For any i , a sequence of best D_i -approximation to P can be found on the fibre of π through P (this includes possibly along some E_j), or else on S .*

Solution: If $i = 0$, then $D_i = F$, so the claim is trivial. If $i = 1$, then $D_i = S + nF$, and against the claim is clearly true. For general i , if P lies on a curve C with $C.D_i = 0$, then clearly C will be a curve of best D_i -approximation to P . However, such a curve C must be linearly equivalent to a nonnegative linear combination of $\{E_j \mid j \neq i\}$, where as usual $E_0 = S$. If C is not S or some E_i , then C must be nef, and so must be a positive linear combination of the D_i . Since $m < n$, Remark 3.12 shows that this is impossible. Thus, if $C.D_i = 0$, then C is either S or some E_i for $i \geq 1$.

Assume that E_i is a leaf with $i > 1$. Then D_i is basepoint free, and $D_i.F = 1$, so if $\phi_i: X \rightarrow \mathbb{P}^n$ is the morphism associated to D_i , then ϕ_i maps the fibre of π through P to a line. By Theorem 2.6, this means that a sequence of best D_i -approximation to P either lies along the fibre of π through P , or else along some irreducible curve C with $C.D_i = 0$, which as noted must either be S or some E_i .

If E_i is not a leaf, then there is some j for which E_j is a leaf and $E_j \succeq E_i$. By Lemma 3.14, we can write $D_i = m_i D_j + \sum_{E_t \succeq E_j} m_t E_t$. If P does not lie on any E_t with $E_t \succeq E_j$, then Theorem 3.3 implies that either the fibre of π through P or S is a curve of best D_i -approximation to P . If P lies on some E_t for $t \neq i$, then E_t is a curve of best D_i -approximation to P because $E_t.D_i = 0$ and some multiple of D_i is basepoint free. If P lies on E_i but not on any other E_t , then as in the proof of Lemma 3.15, we can write $D_i = \alpha F + E$, where α is a positive rational number and E is an effective divisor whose support does not include E_i . Since E_i is a curve of best F -approximation to P , Lemma 3.14 implies that E_i is also a curve of best D_i -approximation to P . In all cases, the claim is proven. ♣

The claim, together with Corollary 3.2, shows that if D is any ample divisor on X , then D is a nonnegative linear combination of the D_i , and so a sequence of best D -approximation to P can be found on the fibre of π which contains P , unless P lies on S . If P lies on S , then S is a curve of best D_i -approximation to P for all i , and so S is also a curve of best D -approximation to P . This concludes the proof of Theorem 3.9. ♣

These same techniques will also prove Conjecture 2.7 for split rational surfaces of Picard rank four.

Theorem 3.17. *Conjecture 2.7 is true for any split rational surface X of Picard rank four.*

Proof: Every such rational surface is a blowup of a Hirzebruch surface, and so X admits a surjective morphism $\pi: X \rightarrow \mathbb{P}^1$ whose generic fibre

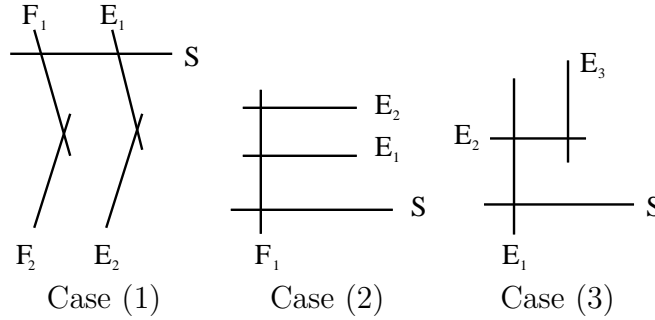


TABLE 1. Configurations of reducible fibres

is irreducible. The Néron-Severi group has rank four, generated by the class F of a fibre of this morphism, the class S of a section, and two more classes, corresponding to irreducible components of reducible fibres, or equivalently, exceptional divisors of the blowing down map to some Hirzebruch surface.

There are three possible configurations of reducible fibres (see Table 1):

- (1) Two reducible fibres, each with two components. These components intersect each other transversely in a single point.
- (2) One reducible fibre, with three components, configured like a letter F, where the vertical component is the one which intersects the section S .
- (3) One reducible fibre, with three components, configured like a letter H, where the leftmost vertical component is the one which intersects the section S .

In the proof, the first case will generate a further subcase, corresponding to \mathbb{P}^2 blown up at three points in general position, and the third case will generate a further subcase corresponding to a double component in the reducible fibre. However, in all cases, the combinatorial description of the reducible fibres, combined with the self-intersection of the section S , will completely determine the intersection product on X . Since our proof relies almost completely on the intersection product, this will suffice to prove Theorem 3.17.

Case (1): The fibration $\pi: X \rightarrow \mathbb{P}^1$ admits two reducible fibres, each having two components. (See Table 1.)

In addition to the classes S and F described above, let E_1 and E_2 be the components of one irreducible fibre, and let F_1 and F_2 be the components of the other. Let E_2 and F_2 be the components which do not intersect S . Since the Picard rank is four, the four classes $S, F, E_2,$

and F_2 are a basis for the vector space $NS(X) \otimes \mathbb{R}$. The intersection matrix for X is:

$$\begin{array}{c|cccc} & F & E_2 & F_2 & S \\ \hline F & 0 & 0 & 0 & 1 \\ E_2 & 0 & -1 & 0 & 0 \\ F_2 & 0 & 0 & -1 & 0 \\ S & 1 & 0 & 0 & -n \end{array}$$

where $n = -S^2$ is a positive integer.

The case $n > 2$ is covered by Theorem 3.5. We first treat the case $n = 2$.

Claim 3.18. *The effective cone of X is generated by the classes of E_i , F_i , and S , and the nef cone of X is generated by the classes of F , $D_2 = 2F + S$, $D_1 = 2F - E_2 + S$, $D'_1 = 2F - F_2 + S$, and $D_0 = 2F - E_2 - F_2 + S$.*

Proof: A straightforward calculation shows that the two cones in the claim are dual to one another, and the first cone is clearly contained in the effective cone, so it suffices to show that the interior of the cone generated by F , D_i and D'_1 is contained in the ample cone. We will use the Nakai-Moishezon Criterion again, and note that the intersection properties of the given five divisors are as follows:

$$\begin{array}{c|ccccc} & F & D_2 & D_1 & D'_1 & D_0 \\ \hline F & 0 & 1 & 1 & 1 & 1 \\ D_2 & 1 & 2 & 2 & 2 & 2 \\ D_1 & 1 & 2 & 1 & 2 & 1 \\ D'_1 & 1 & 2 & 2 & 1 & 1 \\ D_0 & 1 & 2 & 1 & 1 & 0 \end{array}$$

Each of these five divisors are basepoint free: F and D_0 correspond to morphisms to \mathbb{P}^1 , D_1 and D'_1 correspond to birational maps to \mathbb{P}^2 , and $D_2 = D_1 + E_2 = D'_1 + F_2$. It is therefore clear that any positive linear combination of these five divisors has positive self-intersection, and intersects any irreducible curve C positively, so by the Nakai-Moishezon Criterion, must be ample. The claim is therefore proven. ♣

We will now prove Case (1) of Theorem 3.17 for the case $n = 2$. Let $P \in X(k)$ be any rational point — we first assume that P does not lie on S , E_i , or F_i . Let $\psi: X \rightarrow \mathbb{P}^1$ be the morphism corresponding to D_0 . A sequence of best D_0 -approximation to P clearly lies along the fibre of ψ through P , while a sequence of best D -approximation to P for any positive linear combination D of the other four generators lies along the component of π through P . We therefore define two subcones of the nef cone of X : subcone A , generated by F , D_2 , D_1 , D'_1 , and $D_0 + F$, and

subcone B , generated by D_0 , $D_0 + F$, D_1 , and D'_1 . For each generator of subcone A , a sequence of best approximation to P lies along the component of F of minimal degree through P . If P does not lie on a reducible fibre or S , then we can conclude from Corollary 3.2 that if D lies in the cone A , then the fibre of π through P is a curve of best D -approximation to P . For each generator of subcone B , a similar argument shows that a sequence of best approximation to P lies along the component of the fibre of ψ through P . Since the union of A and B is the entire nef cone of X , we conclude that Conjecture 2.7 is true for P .

If P does lie on S , E_i , or F_i , then we divide the nef cone into five subcones, one for each of the five curves S , E_i , and F_i . The subcone A_C corresponding to a curve C is the set of nef divisors D for which C has minimal D -degree amongst S , E_i , and F_i . A straightforward calculation shows that generators for these subcones can always be found from the set $\{F, D_0, D_1, D'_1, D_2, F + D_0, F + D_1, F + D'_1, F + D_2 + D_0\}$, and that each divisor in that set intersects each of S , E_i , and F_i in either 0 or 1. It then follows from Corollary 3.2 that for each curve $C \in \{S, E_1, E_2, F_1, F_2\}$, the curve C is a curve of best D -approximation to P for every divisor $D \in A_C$. Since the nef cone is the union of the subcones A_C , we conclude that Conjecture 2.7 is true in case (1) with $n = 2$.

We now move to the case $n = 1$. In this case, X is the blowup of the first Hirzebruch surface H_1 at two points, possibly (*a priori*) infinitely near. Thus, X is also the blowup of \mathbb{P}^2 at three points (again, possibly infinitely near). If any of the three points are infinitely near, then there will be a curve S of self-intersection -2 or less, and a morphism from X to \mathbb{P}^1 of which S is a section, so X is covered by the already-treated case $n \geq 2$. Thus, the three points must be distinct. If they are collinear, then no morphism from X to \mathbb{P}^1 will have more than one reducible section, so we cannot be in Case (1). Thus, X must be isomorphic to the blowup of \mathbb{P}^2 at three points in general position.

Let $\phi: X \rightarrow \mathbb{P}^2$ be the blowing down map. Let L be the divisor class corresponding to the invertible sheaf $\phi^*\mathcal{O}(1)$, and let E_i for $i = 1, 2, 3$ be the three exceptional divisors of ϕ . These four divisors are a basis of the vector space $NS(X) \otimes \mathbb{R}$, and their intersection matrix is as follows:

$$\begin{array}{c|cccc} & L & E_1 & E_2 & E_3 \\ \hline L & 1 & 0 & 0 & 0 \\ E_1 & 0 & -1 & 0 & 0 \\ E_2 & 0 & 0 & -1 & 0 \\ E_3 & 0 & 0 & 0 & -1 \end{array}$$

Claim 3.19. *The effective cone of X is generated by the six curves E_i and $L - E_i - E_j$ for $i \neq j$. The nef cone of X is generated by the five curves L , $L - E_i$, and $F = 2L - E_1 - E_2 - E_3$.*

Proof: This can be found in, for example, [Ma]. ♣

The divisors L and F correspond to birational morphisms to \mathbb{P}^2 , and $L - E_i$ corresponds to a morphism to \mathbb{P}^1 . Thus, each of the divisors is basepoint free, and their intersection matrix is:

	L	$L - E_1$	$L - E_2$	$L - E_3$	F
L	1	1	1	1	2
$L - E_1$	1	0	1	1	1
$L - E_2$	1	1	0	1	1
$L - E_3$	1	1	1	0	1
F	2	1	1	1	1

We now conclude the proof of Conjecture 2.7 in Case (1). Let $P \in X(k)$ be any k -rational point, and let D be any ample divisor on X . Let A_1 be the cone generated by the following divisor classes:

$$(1) \quad L, L - E_1, (L - E_1) + (L - E_2), (L - E_1) + (L - E_3), F$$

By permuting the indices on the E_i in the above list, we obtain two more cones A_2 and A_3 such that the nef cone of X is the union of the subcones A_i . Assume that D lies in A_1 ; by symmetry, it suffices to consider this case.

If P does not lie on E_i or $L - E_i - E_j$ for $i \neq j$, then Theorem 2.6 and Corollary 3.2 imply that the component of $L - E_m$ through P is a curve of best D -approximation to P . (To see this, note that each of the generators of A_1 is basepoint free, contracts no curves through P , and intersects $L - E_1$ either one or zero times.) Thus, if P does not lie on any E_i or $L - E_i - E_j$, then we have proven Conjecture 2.7 for P .

Now say that P does lie on some E_i or $L - E_i - E_j$. The incidence graph of these six divisors is a hexagon: E_i intersects precisely $L - E_i - E_k$ for $k \neq i$, and $L - E_i - E_j$ intersects precisely E_i and E_j . Thus, P either lies on exactly one or exactly two of these six curves.

If P lies only on E_1 , then consider the subcone B_1 of A_1 , generated by:

$$L, L + (L - E_1), (L - E_1) + (L - E_2), (L - E_1) + (L - E_3), F$$

The generator L contracts E_1 , and so E_1 is clearly a curve of best L -approximation to P . The remaining generators are all basepoint free, intersect E_1 once, and contract no curves through P , and so by Theorem 2.6, E_1 is a curve of best approximation to P with respect to

all the generators of B_1 . Corollary 3.2 now implies that for any D in B_1 , E_1 is a curve of best approximation to P .

If D does not lie in B_1 , then D must lie in the cone B'_1 , generated by:

$$L + (L - E_1), L - E_1, (L - E_1) + (L - E_2), (L - E_1) + (L - E_3), F$$

The generator $L - E_1$ contracts the component C of $L - E_1$ through P , and so C is a curve of best $(L - E_1)$ -approximation to P . Each of the other generators is basepoint free, does not contract any curve through P , and intersects C once, so by Theorem 2.6, C is a curve of best approximation to P with respect to all the generators of B'_1 . Corollary 3.2 now implies that C is a curve of best D -approximation to P . In summary, if P lies on E_1 and no other generator of the effective cone of X , then we have proven Conjecture 2.7 for P .

If P lies only on $E_2, E_3, L - E_1 - E_2$, or $L - E_1 - E_3$, then by a similar argument to the previous paragraph, Theorem 2.6 and Corollary 3.2 imply that for any D in A_1 , one of the curves $E_2, E_3, L - E_1 - E_2$, or $L - E_1 - E_3$ will be a curve of best D -approximation to P , and so Conjecture 2.7 is proven for P .

If P lies only on $L - E_2 - E_3$, then define the cone B_{23} to be that generated by:

$$F, L, (L - E_1) + (L - E_2), (L - E_1) + (L - E_3), F + (L - E_1)$$

Since F contracts $L - E_2 - E_3$, it's clear that $L - E_2 - E_3$ is a curve of best F -approximation to P . All the other generators are basepoint free, do not contract any curves through P , and intersect $L - E_2 - E_3$ once. Thus, by Theorem 2.6 and Corollary 3.2, we conclude that if D lies in B_{23} , $L - E_2 - E_3$ is a curve of best D -approximation to P . If D does not lie in B_{23} , then D must lie in B'_{23} , generated by:

$$L - E_1, L, (L - E_1) + (L - E_2), (L - E_1) + (L - E_3), F + (L - E_1)$$

A similar argument to the previous paragraph shows that the component of $L - E_1$ through P is a curve of best D -approximation to P . We conclude that if P lies only on one of the six generators of the effective cone of X , then Conjecture 2.7 is true for P .

It only remains to consider the possibility that P is the intersection of two of the curves E_i or $L - E_i - E_j$. First, assume that $P = E_1 \cap L - E_1 - E_i$ for $i = 2$ or $i = 3$. Let B_{11i} be the cone generated by:

$$L + (L - E_1), L + F, L + (L - E_1) + (L - E_i), (L - E_1) + (L - E_{5-i}), \\ L - E_1, (L - E_1) + (L - E_i), F$$

The last three generators of B_{11i} all contract $L - E_1 - E_i$, and so $L - E_1 - E_i$ is a curve of best approximation to P for each of these

generators. The first four generators are all basepoint free, contract no curves through P , and map $L - E_1 - E_i$ to a line, and so by Theorem 2.6 and Corollary 3.2 we conclude that $L - E_1 - E_2$ is a curve of best D -approximation to P if D lies in B_{11i} .

If D does not lie in B_{11i} , then D must lie in the cone B'_{11i} , generated by:

$$L, L + (L - E_1), L + (L - E_1) + (L - E_i), (L - E_1) + (L - E_{5-i}), L + F$$

(Recall that D is assumed to lie in the cone A_1 , defined above.) The divisor L contracts E_1 , and the other four generators are basepoint free, contract no curves through P , and map E_1 to a line. We conclude from Theorem 2.6 and Corollary 3.2 that E_1 is a curve of best D -approximation to P .

Next, assume that P lies on the intersection of $L - E_1 - E_i$ and E_i for $i = 2$ or $i = 3$. Define the cone B_{1ii} to be the cone generated by:

$$L, L - E_1, (L - E_1) + (L - E_{5-i}), L + (L - E_1) + (L - E_i), L + F, \\ F + (L - E_1) + (L - E_{5-i})$$

All of these classes are basepoint free. The first three contract E_i , and the last three contract no curves through P and map E_i to a line, and so we conclude by Theorem 2.6 and Corollary 3.2 that E_i is a curve of best D -approximation to P for any D in B_{1ii} . Thus, Conjecture 2.7 is true for P if D lies in B_{1ii} .

If D does not lie in B_{1ii} , then D must lie in the cone B'_{1ii} , generated by:

$$F, L - E_1, (L - E_1) + (L - E_i), L + F, L + (L - E_1) + (L - E_i), \\ F + (L - E_1) + (L - E_{5-i})$$

All generators are basepoint free. The first three generators contract $L - E_1 - E_i$, and the last three contract no curves through P but map $L - E_1 - E_i$ to a line. We conclude by Theorem 2.6 and Corollary 3.2 that $L - E_1 - E_i$ is a curve of best D -approximation to P for all D in B'_{1ii} . We conclude that Conjecture 2.7 is true for $P = E_i \cap (L - E_1 - E_i)$ and all D .

Finally, let P be the intersection of E_i and $L - E_i - E_{5-i}$ for $i = 2$ or $i = 3$. Define B_{123} to be the cone generated by:

$$L, L - E_1, (L - E_1) + (L - E_{5-i}), (L - E_1) + (L - E_i), F + L, F + (L - E_1), \\ F + (L - E_1) + (L - E_{5-i})$$

All of these classes are basepoint free. The first three contract E_i , and the last four do not contract any curve through P , and map E_i to a

line. Thus, by Theorem 2.6 and Corollary 3.2, we conclude that E_i is a curve of best D -approximation to P , provided D lies in B_{123} .

If D does not lie in B_{123} , then it must lie in the cone B'_{123} , generated by:

$$F, F + L, (L - E_1) + (L - E_i), F + (L - E_1), F + (L - E_1) + (L - E_{5-i})$$

All five of these are basepoint free. The last four contract no curves through P , and map $L - E_2 - E_3$ to a line, and F contracts $L - E_2 - E_3$ to a point. By Theorem 2.6 and Corollary 3.2, we conclude that $L - E_i - E_j$ is a curve of best D -approximation to P if D lies in B'_{123} . Conjecture 2.7 is thus proven for X .

Case (2): One reducible fibre, with three components, configured like a letter F, where the vertical component is the one which intersects the section S . (See Table 1.)

Let F_1 be the component of the reducible fibre which intersects S , and let E_1 and E_2 be the other two components. Let F be the class of a fibre. These four classes are a basis of $\text{NS}(X)$, with intersection matrix:

	S	E_1	E_2	F
S	$-n$	0	0	1
E_1	0	-1	0	0
E_2	0	0	-1	0
F	1	0	0	0

Claim 3.20. *The curves S , F_1 , E_1 , and E_2 generate the effective cone of X . The nef cone of X is generated by the divisors F , $D_1 = nF + S$, $D_2 = nF + S - E_1$, and $D_3 = nF + S - E_2$.*

Proof: A straightforward calculation shows that the two cones described in the claim are dual to one another. Thus, it suffices to show that F , D_1 , D_2 , and D_3 generate the nef cone of X . To do this, it further suffices to show that every positive linear combination of F , D_1 , D_2 , and D_3 is ample, since the converse inclusion is clear from the effectivity of S , F_1 , E_1 , and E_2 .

The intersections of F and the D_i are tabulated as follows:

	F	D_1	D_2	D_3
F	0	1	1	1
D_1	1	n	n	n
D_2	1	n	$n - 1$	n
D_3	1	n	n	$n - 1$

Since $n \geq 1$, it is clear that any positive linear combination of F and the D_i must have positive self-intersection. It's clear that F is

basepoint free, since it corresponds to a morphism to \mathbb{P}^1 , and similarly each D_i corresponds to a morphism to a cone over a rational normal curve (see Remark 3.21). In particular, each of these four divisors is nef, so by the Nakai-Moishezon criterion for ampleness, we conclude that they must generate the nef cone of X , as desired. ♣

Remark 3.21. Let Y be the cone over a smooth rational normal curve of degree n . It is easy to see that if we blow up Y at the vertex, we obtain the Hirzebruch surface H_n . Moreover, the exceptional curve of the blowup $\pi: H_n \rightarrow Y$ is precisely the unique $(-n)$ -section S' of H_n , and if L is the class of a hyperplane section of Y , then $\pi^*L = S' + nF'$, where F' is a fibre of the morphism $H_n \rightarrow \mathbb{P}^1$.

Our rational surface X admits several birational maps to Hirzebruch surfaces. It is a straightforward calculation to see that each D_i is the pullback of π^*L via one of these maps. For example, we might blow down E_1 and then F_1 to obtain H_n , in which case pulling back π^*L to X gives D_3 . This technique will be used frequently in what follows.

We now prove Conjecture 2.7 for Case (2). We first assume that $n > 1$. Let $P \in X(k)$ be any rational point not lying on S , F_1 , or either E_i . Let C be the (irreducible) component of F through P . Then F contracts C , and all of the other generators of the nef cone map C to a line, and so by Theorem 2.6 and Corollary 3.2, we conclude that C is a curve of best D -approximation to P for any ample divisor D .

If P lies on S but not F_1 , then we divide the nef cone of X into two subcones: a cone A generated by $D_1, D_2, D_3, D_1 + F, D_2 + F$, and $D_3 + F$, and a cone B generated by $D_1 + F, D_2 + F, D_3 + F$, and F . The classes D_i each contract S , and $F + D_i$ is basepoint free, contracts no curves through P but maps S to a line, and so for each of these six divisor classes, S is a curve of best approximation to P . Thus, by Corollary 3.2, Conjecture 2.7 is true for P with respect to any ample divisor in A . On the other hand, $F + D_i$ also maps the component C of F through P to a line, and F contracts C , so similar reasoning shows that Conjecture 2.7 is true for P with respect to any ample divisor in B . We conclude that Conjecture 2.7 is true for P .

If P lies on E_i but not F_1 , or if P lies on F_1 but not S or either E_i , then let C be the unique irreducible component of F through P (that is, either F_1 or E_i). Every generator of the nef cone either contracts C or else maps C to a line and contracts no curves through P , and so Theorem 2.6 and Corollary 3.2 imply as usual that C is a curve of best D -approximation to P for any ample divisor D .

If $P = S \cap F_1$, then consider the cone A' generated by:

$$F, F + D_1, D_2, D_3$$

Note that all four of these classes are basepoint free. All but $F + D_1$ contract F_1 , and $F + D_1$ maps F_1 to a line and contracts no curve through P . We conclude from Theorem 2.6 and Corollary 3.2 that F_1 is a curve of best D -approximation to P for any D in A' . If D is not in A' , then D must lie in B' , generated by:

$$D_1, F + D_1, D_2, D_3$$

Every generator is basepoint free, and all but $F + D_1$ contracts S . The class $F + D_1$ maps S to a line, and contracts no curves through P . We conclude by Theorem 2.6 and Corollary 3.2 that S is a curve of best D -approximation to P for every D in B' . Thus, Conjecture 2.7 is true for P .

Finally, if $P = F_1 \cap E_i$ for some i , then consider the cone \tilde{A} generated by F , $D_1 + D_{i+1}$, D_2 , and D_3 . Each generator is basepoint free, and all but $D_1 + D_{i+1}$ contract F_1 . Furthermore, the class $D_1 + D_{i+1}$ maps F_1 to a line and contracts no curves through P . Theorem 2.6 and Corollary 3.2 now imply that F_1 is a curve of best D -approximation to P for any D in \tilde{A} .

If D does not lie in \tilde{A} , then it must lie in \tilde{B} , generated by F , D_1 , $D_1 + D_{i+1}$, and D_{4-i} . Each generator is basepoint free, and all but $D_1 + D_{i+1}$ contract E_i . Furthermore, the class $D_1 + D_{i+1}$ maps E_i to a line and contracts no curves through P . Theorem 2.6 and Corollary 3.2 now imply that E_i is a curve of best D -approximation to P for any D in \tilde{B} . We conclude that Conjecture 2.7 is true for X in the case that $n > 1$.

(Note that if $n > 3$, this result also follows immediately from Theorem 3.9.)

If $n = 1$, then $D_2^2 = D_3^2 = 0$, so these both correspond to morphisms to \mathbb{P}^1 . In fact, if we cannot describe X with $n > 1$, then X must be a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at two (possibly infinitely near) points. In fact, the points cannot be infinitely near, since otherwise the exceptional divisor of the first blowup would have self-intersection -2 , and we could choose it to be the section S , with $n = 2 > 1$. And if the configuration of exceptional divisors is to be as in Case (2), the two blown up points must lie on the same fibre of at least one of the canonical projections $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

In other words, we have shown that X must be isomorphic to \mathbb{P}^2 blown up at three different points on a straight line. The exceptional

curves of this blowup are S , E_1 , and E_2 , while F_1 is the strict transform of the line joining the blown up points. The divisors D_2 and D_3 correspond to the maps $\pi_i: X \rightarrow \mathbb{P}^1$ ($i = 2, 3$) which factor through the blowup of \mathbb{P}^2 at a single point, while F corresponds to the third map $\pi_1: X \rightarrow \mathbb{P}^1$, induced in the same way. The divisor D_1 corresponds to the map $\phi: X \rightarrow \mathbb{P}^2$ that blows down S , E_1 , and E_2 .

Let $P \in X(k)$ be a rational point which does not lie on S , F_1 , or either E_i , and let D be an ample divisor on X . By symmetry, we may assume that $D.F \leq D.D_i$ for $i = 2, 3$, since F , D_2 , and D_3 are all conjugate under the automorphism group of X . In that case, D lies in the cone A generated by:

$$F, D_1, F + D_2, F + D_3, F + D_2 + D_3$$

All these divisors are basepoint free. The last four contract no curves through P and map the component C of F through P to a line, and F contracts C . Thus, by Theorem 2.6 and Corollary 3.2, we conclude that C is a curve of best D -approximation to P , and that Conjecture 2.7 is true for P .

If P lies on one or more of S , E_1 , E_2 , or F_1 , then we must subdivide A further. If P lies on S but not F_1 , then assume that D lies in the cone A_S generated by:

$$D_1, F + D_1, F + D_2, F + D_3, F + D_2 + D_3$$

Each of these divisors is basepoint free. The last four contract no curves through P , and map S to a line, and D_1 contracts S , so by Theorem 2.6 and Corollary 3.2, we conclude that S is a curve of best D -approximation to P . If D does not lie in A_S , then D must lie in A'_S , generated by:

$$F, F + D_1, F + D_2, F + D_3$$

All of these divisors are basepoint free. The last four contract no curves through P and map the component C of F through P to a line, and D_1 contracts C . We conclude from Theorem 2.6 and Corollary 3.2 that C is a curve of best D -approximation to P . In all cases, if P lies on S but not F_1 , then Conjecture 2.7 is true for P .

If P lies on exactly one of E_1 , E_2 , or F_1 , then a quick check (using Theorem 2.6 and Corollary 3.2) shows that E_i is a curve of best D -approximation for every D in the cone A . The same is true if $P = S \cap F_1$.

The only remaining case is thus $P = F_1 \cap E_i$. Consider the cone A_i generated by:

$$F, D_1, F + D_{4-i}, F + D_1 + D_{i+1}, F + D_1 + D_2 + D_3$$

All these divisors are basepoint free. The first three generators contract E_i , while the last two contract no curves through P but map E_i to a line. By Theorem 2.6 and Corollary 3.2, we conclude that E_i is a curve of best D -approximation to P for any D in A_i . If D does not lie in A_i , then D must lie in the cone A'_i , generated by:

$$F, F + D_2, F + D_3, F + D_2 + D_3, F + D_1 + D_2, F + D_1 + D_2 + D_3$$

All of these divisors are basepoint free. The first four contract F_1 , and the last two contract no curves through P but map F_1 to a line, so we conclude by Theorem 2.6 and Corollary 3.2 that F_1 is a curve of best D -approximation to P .

The sum of all these calculations is that Conjecture 2.7 is true for X in Case (2).

Case (3): One reducible fibre, with three components, configured like a letter H, where the leftmost vertical component is the one which intersects the section S . (See Table 1.)

We first deal with the case in which the reducible fibre has no multiple components. Let E_1 be the component of the reducible fibre which intersects the section S . Let E_2 be the component which intersects both other components, and let E_3 be the remaining component. Let $F = E_1 + E_2 + E_3$ be the class of a fibre. These four classes are a basis of $\text{NS}(X)$, with intersection matrix:

	S	E_2	E_3	F
S	$-n$	0	0	1
E_2	0	-2	1	0
E_3	0	1	-1	0
F	1	0	0	0

Claim 3.22. *The curves S , E_1 , E_2 , and E_3 generate the effective cone of X . The nef cone of X is generated by the divisors F , $D_1 = nF + S - E_2 - 2E_3$, $D_2 = nF + S - E_2 - E_3$, and $D_3 = nF + S$.*

Proof: A straightforward calculation shows that the two cones described in the claim are dual to one another. Thus, it suffices to show that F , D_1 , D_2 , and D_3 generate the nef cone of X . To do this, it further suffices to show that every positive linear combination of F , D_1 , D_2 , and D_3 is ample, since the converse inclusion is clear from the effectivity of S , F_1 , E_1 , and E_2 .

The intersections of F and the D_i are tabulated as follows:

	F	D_1	D_2	D_3
F	0	1	1	1
D_1	1	$n - 2$	$n - 1$	n
D_2	1	$n - 1$	$n - 1$	n
D_3	1	n	n	n

We first note that n can be chosen to be at least two, since if $n = 0$ or $n = 1$, then X is a blowup of \mathbb{P}^2 at two infinitely near points, and is therefore also a blowup of the second Hirzebruch surface H_2 , in which case we can take $n = 2$.

Now assume that $n \geq 2$. It's clear that F and each of the D_i are basepoint free: F corresponds to a morphism to \mathbb{P}^1 , D_1 corresponds to a morphism to a cone over a smooth rational curve of degree $n - 2$ (if $n = 2$ we take this cone to be \mathbb{P}^1), D_2 corresponds to a morphism to a cone over a smooth rational curve of degree $n - 1$, and D_3 corresponds to a morphism to a cone over a smooth rational curve of degree n (see Remark 3.21). For each D_i , the ruling of the cone pulls back to F . In particular, every positive linear combination of F and the D_i has positive intersection with every curve on X , and has positive self-intersection. Therefore, by the Nakai-Moishezon criterion, every such positive linear combination is ample, and we have proven the claim. ♣

We now prove Conjecture 2.7 for X . Let $P \in X(k)$ be any rational point. The case $n > 3$ is immediate from Theorem 3.9.

Assume $n = 3$, and let $P \in X(k)$ be any k -rational point. If P does not lie on S or any E_i , then every generator of the nef cone either contracts the component C of F through P , or else contracts no curves through P and maps C to a line. Thus, Theorem 2.6 and Corollary 3.2 imply that for any ample divisor D , C is a curve of best D -approximation to P .

If P lies on S but not E_1 , then consider the cone A_S , generated by:

$$F, F + D_1, F + D_2, F + D_3$$

Each divisor is basepoint free. The last three divisors contract no curves through P , but map the component C of F through P to a line, while F contracts C . By Theorem 2.6 and Corollary 3.2, we conclude that C is a curve of best D -approximation to P for any D in A_S . If D does not lie in A_S , then it must lie in A'_S , generated by:

$$D_1, F + D_1, D_2, F + D_2, D_3, F + D_3$$

Each D_i contracts S , while $F + D_i$ contracts no curves through P and maps S to a line. Thus, by Theorem 2.6 and Corollary 3.2, we

conclude that S is a curve of best D -approximation to P for every $D \in A'_G$. Conjecture 2.7 is thus proven for P .

If P lies on exactly one of the E_i but not S , then for each generator G of the nef cone, we see that either G contracts E_i or else maps E_i to a line and contracts no curves through P . Theorem 2.6 and Corollary 3.2 now imply that E_i is a curve of best D -approximation to P .

If $P = S \cap E_1$, then consider the cone A_1 , generated by:

$$F, D_1, D_2, F + D_3$$

The first three all contract E_1 , and the last contracts no curves through P but maps E_1 to a line. By Theorem 2.6 and Corollary 3.2, we conclude that E_1 is a curve of best D -approximation to P for all $P \in A_1$. If D does not lie in A_1 , then D must lie in A'_1 , generated by:

$$D_1, D_2, D_3, F + D_3$$

The first three curves all contract S , while the last contracts no curves through P but maps S to a line. Theorem 2.6 and Corollary 3.2 now imply that S is a curve of best D -approximation to P for any D in A'_1 .

If $P = E_1 \cap E_2$, then consider the cone A_2 , generated by:

$$F, D_1, D_3, D_2 + D_3$$

The first three classes all contract E_2 , and the last contracts no curves through P but maps E_2 to a line. We conclude by Theorem 2.6 and Corollary 3.2 that E_2 is a curve of best D -approximation to P for all D in A_2 . If D does not lie in A_2 , then D must lie in A'_2 , generated by:

$$F, D_1, D_2, D_2 + D_3$$

The first three contract E_1 , and the last contracts no curve through P but maps E_1 to a line. By Theorem 2.6 and Corollary 3.2, we conclude that E_1 is a curve of best D -approximation to P for all D in A'_2 . Conjecture 2.7 is thus proven for P .

Finally, if $P = E_2 \cap E_3$, then consider the cone A_3 , generated by:

$$F, D_2, D_3, D_1 + D_2$$

The first three all contract E_3 , while the last contracts no curves through P but maps E_3 to a line. We conclude by Theorem 2.6 and Corollary 3.2 that E_3 is a curve of best D -approximation to P for any D in A_3 . If D does not lie in A_3 , then D must lie in A'_3 , generated by:

$$F, D_1, D_3, D_2 + D_3$$

The first three contract E_2 , while that last contracts no curves through P but maps E_2 to a line. Theorem 2.6 and Corollary 3.2 now imply that E_2 is a curve of best D -approximation to P for all D in A'_3 . We

conclude that Conjecture 2.7 is true for this P . Since this exhausts the possibilities for P , we conclude that Conjecture 2.7 is proven for X in Case (3) with $n = 3$.

If $n = 2$, then both F and D_1 correspond to maps to \mathbb{P}^1 , so that X is realized as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at two infinitely near points. Let P be any k -rational point on X which does not lie on S or any E_i . Let C_1 and C_F be the components of D_1 and F , respectively, through P . Consider the cone A , generated by:

$$F, F + D_1, D_2, D_3, D_1 + D_3$$

All of these classes are basepoint free, and all but the last either contract C_F or else contract no curves through P and map C_F to a line. Thus, if we can show that C_F is a curve of best $(D_1 + D_3)$ -approximation to P , then Theorem 2.6 and Corollary 3.2 will imply that C_F is a curve of best D -approximation to P for any D in A .

To prove this, note that $C_F.D_1 = C_F.D_3 = 1$, so that a sequence of best D_1 - or D_3 -approximation to P along $C.F$ has constant of approximation equal to 1. Thus, by Theorem 3.1, since D_1 and D_3 are both basepoint free, it follows that any sequence with a better constant of $(D_1 + D_3)$ -approximation to P than $1 + 1 = 2$ must have all but finitely many points contained in a union of curves through P which are contracted by D_1 or D_3 . The only such curve is C_1 , which is only contracted by D_1 . Since $C_1.(D_1 + D_3) = 2$, it follows that no sequence of points on C_1 can have a constant of $(D_1 + D_3)$ -approximation to P of better than 2. Thus, C_F is a curve of best $(D_1 + D_3)$ -approximation to P , and since $C_1.(D_1 + D_3) = 2$, we see that C_1 is also a curve of best $(D_1 + D_3)$ -approximation to P . (Recall that P is assumed not to lie on S or any E_i .) We conclude that C_F is a curve of best D -approximation to P for any D in A .

If D does not lie in A , then it must lie in the cone B , generated by:

$$D_1, F + D_1, D_2, D_1 + D_3$$

Each divisor is basepoint free, and the first three each either contract C_1 or else contract no curves through P and map C_1 to a line, and we have already seen that C_1 is a curve of best $(D_1 + D_3)$ -approximation to P . Thus, by Corollary 3.2, we conclude that C_1 is a curve of best D -approximation to P for any D in B . It follows that Conjecture 2.7 is true for P .

If P lies on S but not any E_i , then for any D in the cone B , it is easy to check that S is a curve of best D -approximation to P . If D lies in A , then consider the cone A_S , generated by:

$$D_1 + D_3, D_2, D_3, F + D_1, F + D_2, F + D_3$$

The first three of these contract S , and the last three contract no curves through P but map S to a line. By Theorem 2.6 and Corollary 3.2, it follows that S is a curve of best D -approximation to P for all D in A_S . If D does not lie in A_S or B , then it must lie in the cone A'_S , generated by:

$$F, F + D_1, F + D_2, F + D_3$$

The first of these contracts C_F , and the last three contract no curves through P but map C_F to a line. By Theorem 2.6 and Corollary 3.2, it follows that C_F is a curve of best D -approximation to P for all D in A'_S . We conclude that Conjecture 2.7 is true for P .

Next, assume that P lies on a curve C which is either E_1 or E_2 , and assume that P does not lie on S or E_3 , and is not the point $E_1 \cap E_2$. It is easy to check that each generator of the nef cone of X either contracts C or else contracts no curves through P and maps C to a line. We conclude that for any ample divisor D , the curve C is a curve of best D -approximation to P .

If P lies on E_3 but not E_2 , then it is easy to check that every generator of the cone A either contracts E_3 or else contracts no curves through P but maps E_3 to a line. Thus, if D lies in A , then E_3 is a curve of best D -approximation to P . If D does not lie in A , then consider the cone A_3 , generated by:

$$D_2, D_1 + D_3, F + D_1, D_1 + D_2, 2D_1 + D_3$$

It is clear that E_3 is contracted by D_2 and mapped to a line by the next three divisors (which contract no curves through P). Moreover, E_3 is a curve of best D_1 -approximation to P , and therefore is also a curve of best $2D_1$ -approximation to P , and by Theorem 3.3, since $E_3 \cdot D_3 = 0$, it follows that E_3 is also a curve of best $(2D_1 + D_3)$ -approximation to P . By Corollary 3.2, it follows that E_3 is a curve of best D -approximation to P for any D in A_3 . If D does not lie in either A_3 or A , then it must lie in B_3 , generated by:

$$D_1, F + D_1, D_1 + D_2, 2D_1 + D_3$$

The first of these contracts C_1 , the next two contract no curves through P but map C_1 to a line, and since $(2D_1 + D_3) \cdot C_1 = 2 = (2D_1 + D_3) \cdot E_3$, it follows that C_1 is a curve of best $(2D_1 + D_3)$ -approximation to P as well. Thus, by Corollary 3.2, we deduce that C_1 is a curve of best D -approximation to P for any D in B_3 , and hence for any D in the nef cone.

This leaves just three points P to check. If $P = S \cap E_1$, then consider the cone A_1 , generated by:

$$F + D_3, D_1, D_2, D_3$$

All of these are basepoint free, and all but the first of them contract S . Since $F + D_3$ contracts no curves through P but maps S to a line, we conclude by Theorem 2.6 and Corollary 3.2 that S is a curve of best D -approximation to P for all D in A_1 . If D does not lie in A_1 , then D must lie in B_1 , generated by:

$$F, D_1, D_2, F + D_3$$

The first three of these contract E_1 , and the last maps E_1 to a line but does not contract any curve through P . By Theorem 2.6 and Corollary 3.2, we conclude that E_1 is a curve of best D -approximation to P for any D in B_1 , and hence that Conjecture 2.7 is true for P .

If $P = E_1 \cap E_2$, then consider the cone A_2 , generated by:

$$F, D_1, D_2, D_2 + D_3$$

The first three of these contract E_1 , and the last contracts no curves through P but maps E_1 to a line. We conclude by Theorem 2.6 and Corollary 3.2 that E_1 is a curve of best D -approximation to P for all D in A_2 . If D does not lie in A_2 , then D must lie in B_2 , generated by:

$$F, D_1, D_3, D_2 + D_3$$

By an exactly similar argument to that used for A_2 , we conclude that E_2 is a curve of best D -approximation to P for any D in B_2 , and therefore that Conjecture 2.7 is true for P .

Finally, we consider the case that $P = E_2 \cap E_3$. Let A'_2 be the cone generated by:

$$F, D_1, D_3, D_1 + D_2$$

The first three divisors contract E_2 , and the last contracts no curves through P but maps E_2 to a line. Theorem 2.6 and Corollary 3.2 thus imply that E_2 is a curve of best D -approximation to P for any D in A'_2 . If D does not lie in A'_2 , then it must lie in B'_2 , generated by:

$$F, D_2, D_3, D_1 + D_2$$

Again, each of the first three divisors contracts E_3 , while the last contracts no curves through P but maps E_3 to a line. We conclude from Theorem 2.6 and Corollary 3.2 that E_3 is a curve of best D -approximation to P for any D in B'_2 , and therefore that Conjecture 2.7 is true for P . This completes the proof of Conjecture 2.7 for Case (3), provided that the reducible fibre has no multiple components.

To finish the proof, it remains only to consider the case of a multiple component in the reducible fibre. The only way this can occur is if the crossbar E_2 of the H has multiplicity two in the fibre.

Claim 3.23. *The effective cone of X is generated by the classes of S , E_1 , E_2 , and E_3 . The nef cone of X is generated by the classes of F , $D_1 = S + nF$, $D_2 = 2S + 2nF - 2E_2 - E_3$, and $D_3 = S + nF - E_2 - E_3$.*

Proof of claim: A simple calculation shows that the two cones are dual, so to prove the claim it suffices to show that every positive linear combination of F , D_1 , D_2 , and D_3 is ample. The classes D_1 and D_3 are basepoint free, since they correspond to morphisms to cones (see Remark 3.21), and F is also basepoint free because it corresponds to a morphism to \mathbb{P}^1 . Finally, note that D_2 is basepoint free because it is linearly equivalent both to $2D_3 + E_3$ and $D_1 + S + (n-1)F + E_1$.

Thus, to prove the claim, we may invoke the Nakai-Moishezon Criterion for ampleness so that it suffices to show that every positive linear combination of F , D_1 , D_2 , and D_3 has positive self-intersection. These four divisors have intersection numbers as follows:

	F	D_1	D_2	D_3
F	0	1	2	1
D_1	1	n	$2n$	n
D_2	2	$2n$	$4n - 2$	$2n - 1$
D_3	1	n	$2n - 1$	$n - 1$

Thus, provided that we choose $n \geq 1$ – which we may do without loss of generality – the claim is proven. ♣

We now prove Conjecture 2.7 in the case that the reducible fibre has a multiple component. First, assume that P does not lie on S or the reducible fibre. The component C of F through P is, by Theorem 2.6, a curve of best D_1 - and D_3 -approximation to P (since these divisors contract no curves through P), and it's clear that C is a curve of best F -approximation to P . Finally, if we write $D_2 = 2D_3 + E_3$, then Corollary 3.3 implies that C is a curve of best D_2 -approximation to P . Thus, by Corollary 3.2, we see that C is a curve of best D -approximation to P for all ample divisors D .

Next, assume that P lies on S but not any E_i , and consider the cone A_S , generated by:

$$D_1, D_2, D_3, F + D_1, F + D_3, 2F + D_2$$

All of these divisors are basepoint free. The first three contract S , the next two contract no curves through P and map S to a line, so by Theorem 2.6, S is a curve of best approximation to P with respect to any of these five divisors. Moreover, S is a curve of best F -approximation to P , and since $D_2.S = 0$ and D_2 is basepoint free, it follows from Corollary 3.3 that S is also a curve of best $(2F + D_2)$ -approximation

to P . Thus, by Corollary 3.2, it follows that S is a curve of best D -approximation to P for all D in A_S . If D does not lie in A_S , then it must lie in the cone B_S , generated by:

$$F, F + D_1, F + D_3, 2F + D_2$$

Since F contracts the component C of F through P , we see that C is a curve of best F -approximation to P . Furthermore, $F + D_1$ and $F + D_3$ both contract no curves through P but map C to a line, so C is a curve of best approximation to P with respect to those two divisors as well. Finally, since $C.(2F + D_2) = 2 = S.(2F + D_2)$, the fact that S is a curve of best $(2F + D_2)$ -approximation to P implies immediately that C is also a curve of best $(2F + D_2)$ -approximation to P . By Corollary 3.2, we conclude that Conjecture 2.7 is true for P .

If P lies on exactly one E_i but not S , then F and D_j for $j \neq i$ contract E_i , and D_i contracts no curves through P but maps E_i to a line. Thus, by Theorem 2.6 and Corollary 3.2, E_i is a curve of best D -approximation to P for any ample divisor D .

If $P = S \cap E_1$, consider the cone A_1 , generated by:

$$F + D_1, D_2, D_3, F$$

The last three divisors contract E_1 , and the first contracts no curves through P but maps E_1 to a line. We conclude by Theorem 2.6 and Corollary 3.2 that E_1 is a curve of best D -approximation to P for any D in A_1 . If D does not lie in A_1 , then it must lie in B_1 , generated by:

$$D_1, D_2, D_3, F + D_1$$

The first three divisors contract S , while the last contracts no curves through P but maps S to a line. By Theorem 2.6 and Corollary 3.2, it follows that S is a curve of best D -approximation to P for any D in B_1 . Conjecture 2.7 is therefore proven for P .

If $P = E_1 \cap E_2$, then consider the cone A_2 , generated by:

$$F, D_1, D_3, D_1 + D_2$$

The first three contract E_2 , and the last contracts no curves through P but maps E_2 to a line. Theorem 2.6 and Corollary 3.2 now imply that E_2 is a curve of best D -approximation to P for any D in A_2 . If D does not lie in A_2 , then it must lie in B_2 , generated by:

$$F, D_2, D_3, D_1 + D_2$$

The first three contract E_1 , while the last contracts no curves through P but maps E_1 to a line. Theorem 2.6 and Corollary 3.2 now imply that E_1 is a curve of best D -approximation to P for any D in B_2 . Conjecture 2.7 is therefore proven for P .

Finally, assume that $P = E_2 \cap E_3$. Assume that D lies in the cone A_3 , generated by:

$$F, D_1, D_2, D_2 + D_3$$

The first three of these contract E_3 , and the last contracts no curves through P but maps E_3 to a line. By Theorem 2.6 and Corollary 3.2, it follows that E_3 is a curve of best D -approximation to P for any D in A_3 . If D does not lie in A_3 , then it must lie in B_3 , generated by:

$$F, D_1, D_3, D_2 + D_3$$

The first three contract E_2 , and the last contracts no curves through P but maps E_2 to a line. By Theorem 2.6 and Corollary 3.2, we conclude that E_2 is a curve of best D -approximation to P for any D in B_3 . Conjecture 2.7 is therefore proven for X , for Case (3), and for any split rational surface of Picard rank at most four. ♣

When the Picard rank is larger than four, the number of different cases to consider becomes much larger, so we will prove only a few cases, hopefully representative of the general flavour. We first turn our attention to the blowup of \mathbb{P}^2 at four k -rational points in general position. The surface X admits a morphism $\pi_1: X \rightarrow \mathbb{P}^1$ whose fibres are the strict transforms of the conics through the four blown up points, so that in particular $\pi_1^*\mathcal{O}(1) = 2L - E_1 - E_2 - E_3 - E_4$. It also admits a morphism $\pi_2: X \rightarrow \mathbb{P}^2$, which is the blowing down map. Let E_1, E_2, E_3 , and E_4 be the four exceptional divisors of π_2 , and let $L = \pi_2^*\mathcal{O}(1)$ be the pullback of a line in \mathbb{P}^2 .

Theorem 3.24. *Let X be the blowup of \mathbb{P}^2 at four k -rational points in general position, and let $P \in X(k)$ be any k -rational point. Then a sequence of best approximation to P can be chosen to lie as a subset of the rational curve of minimal degree through P .*

Proof: The geometry of X is well understood (see for instance Example 2.1.2 of [Tsch], or [Ma]). The nef cone of X is generated by the ten classes $L, L_i = L - E_i, D = 2L - E_1 - E_2 - E_3 - E_4$, and $D_i = D + E_i$ for $i = 1, 2, 3, 4$. Each of these is basepoint free: D and each L_i is the fibre of a morphism to \mathbb{P}^1 , while L and each D_i corresponds to a birational morphism to \mathbb{P}^2 , each one blowing down four pairwise disjoint smooth rational curves. There are exactly ten (-1) -curves on X , namely E_i and $L - E_i - E_j$ for $i = 1, 2, 3, 4$ and $i \neq j$.

We have the following table of intersection numbers:

	L	L_1	L_2	L_3	L_4	D_1	D_2	D_3	D_4	D
L	1	1	1	1	1	2	2	2	2	2
L_1	1	0	1	1	1	2	1	1	1	1
L_2	1	1	0	1	1	1	2	1	1	1
L_3	1	1	1	0	1	1	1	2	1	1
L_4	1	1	1	1	0	1	1	1	2	1
D_1	2	2	1	1	1	1	2	2	2	1
D_2	2	1	2	1	1	2	1	2	2	1
D_3	2	1	1	2	1	2	2	1	2	1
D_4	2	1	1	1	2	2	2	2	1	1
D	2	1	1	1	1	1	1	1	1	0

Let $P \in X(k)$ be any k -rational point which does not lie on a (-1) -curve. Define B to be the cone of nef divisors A such that $A.D \leq A.L_i$ for all i . Similarly, for each i between 1 and 4, define the cone B_i as the cone of all nef divisors A such that $A.L_i \leq A.D$ and $A.L_i \leq A.L_j$ for $j \neq i$. A short calculation shows that B is generated by the ten classes:

$$D, D + L, D_i, D + L_i$$

for $i = 1, 2, 3, 4$, and that B_i is generated by the ten classes:

$$L, D + L, L_i, L_i + L_j, D + L_i, D_j$$

for $j \neq i$.

Consider the cone B_i . Let C_i be the component of L_i through P , and let C_D be the component of D through P . Each generator of B_i except $L + D$ either contracts C_i , or else it contracts no curves through P and maps C_i to a line. Thus, by Theorem 2.6, C_i is a curve of best approximation to P with respect to all the generators of B_i except possibly $L + D$. For $L + D$, note that $C_i.L = C_i.D = 1$, so that a sequence of best L - or D -approximation to P along C_i has constant of approximation equal to 1. Since L and D are basepoint free, it follows from Theorem 3.1 and Theorem 2.6 that any sequence with a better constant of $(L + D)$ -approximation to P than $1 + 1 = 2$ must have all but finitely many points contained in a union of curves through P which are contracted by L or D . The only such curve is C_D , which is contracted by D but has L -degree 2, and so a sequence of best $(L + D)$ -approximation to P along C_D also has constant of approximation equal to 2. Thus, we conclude that both C_i and C_D are curves of best $(L + D)$ -approximation to P . By Corollary 3.2, we conclude that Conjecture 2.7 is true for any divisor A in B_i .

Now consider the cone B . Each generator of B except $L + D$ either contracts C_D , or else contracts no curves through P and maps C_D to a line. Thus, by Theorem 2.6, C_D is a curve of best approximation to P with respect to all the generators of B , except possibly $L + D$. We have already shown that C_D is a curve of best $(L + D)$ -approximation to P , so by Corollary 3.2, it follows that Conjecture 2.7 is true for any ample divisor A in B , and thus for any ample divisor A .

There remains the possibility that P might lie on some (-1) -curve. Assume first that P lies on exactly one (-1) -curve. By applying a suitable automorphism of X , we may assume without loss of generality that this curve is E_i . It is easy to check that any generator of the cone B_j ($j \neq i$) either contracts E_i or else contracts no curves through P and maps E_i to a line. Thus, by Theorem 2.6 and Corollary 3.2, it follows that E_i is a curve of best A -approximation to P for any A in B_j for $j \neq i$.

If A lies in the cone B_i , then consider the subcone M_i of B_i generated by:

$$L + L_i, L_i, L_i + L_j, D_j, D + L_i$$

Each generator is basepoint free, and either contracts C_i or else contracts no curves through P and maps C_i to a line. Thus, by Theorem 2.6 and Corollary 3.2, we conclude that C_i is a curve of best A -approximation to P for any A in M_i . If A lies in B_i but not M_i , then it must lie in N_i , generated by:

$$L, L + L_i, L_i + L_j, D_j, D + L$$

Each generator is basepoint free, and either contracts E_i or else contracts no curves through P and maps E_i to a line. Thus, by Theorem 2.6 and Corollary 3.2, we conclude that E_i is a curve of best A -approximation to P for any A in N_i , and therefore for any A in B_i .

The only other possibility is that A lies in the cone B . Consider the cone J_i , generated by:

$$D_i, D_j, L + D, L_j + D, D_i + D$$

where $j \neq i$. All divisors are basepoint free, and either contract E_i or else contract no curves through P and map E_i to a point. By Theorem 2.6 and Corollary 3.2, we conclude that E_i is a curve of best A -approximation to P if A lies in J_i . If A lies in B but not in J_i , then A is in the cone J'_i , generated by:

$$D, D_j, L_i + D, L_j + D, D_i + D$$

where $j \neq i$. All divisors are basepoint free, and either contract D or else contract no curves through P and map D to a point. By Theorem 2.6 and Corollary 3.2, we conclude that D is a curve of best A -approximation to P if A lies in J'_i . This concludes the proof of Conjecture 2.7 for P lying on a single (-1) -curve.

Finally, assume that $P = E_i \cap L - E_i - E_j$ for some i and j . By symmetry, we may assume that $i = 1$ and $j = 2$. Let A be any element of the nef cone. Consider the cone M_1 , generated by:

$$L, L + L_1, L_2, L_3, L_4, L_1 + L_3, L_1 + L_4, D_1, D_2, D_3 + L_4, D_4 + L_3, \\ D + L_3, D + L_4, D + D_1$$

Each generator of M_1 is basepoint free, and either contracts E_1 or else contracts no curves through P and maps E_1 to a line. Theorem 2.6 and Corollary 3.2 now imply that E_1 is a curve of best A -approximation to P for any A in M_1 . If A does not lie in M_1 , then it must lie in the cone N_1 , generated by:

$$L_1, L_2, L + L_1, L_1 + L_3, L_1 + L_4, D_2, D_3, D_4, D, D_3 + L_4, D_4 + L_3, D + L_3, \\ D + L_4, D + D_1$$

Each generator of N_1 is basepoint free, and either contracts $L - E_1 - E_2$ or else contracts no curves through P and maps $L - E_1 - E_2$ to a line. By Theorem 2.6 and Corollary 3.2, we conclude that $L - E_1 - E_2$ is a curve of best A -approximation to P for any A in N_1 , and hence for any ample divisor A .

This concludes the proof of Conjecture 2.7 for X . ♣

Theorem 3.25. *Let X be the blowup of \mathbb{P}^2 at five k -rational points in general position. Conjecture 2.7 is true for X .*

Proof: The geometry of X is well understood (see, for example, [Ma] or [Dr]). The surface X admits a blowing-down map $\pi: X \rightarrow \mathbb{P}^2$. Let L be the class of the preimage of a line, and let E_1, E_2, E_3, E_4 , and E_5 be the exceptional curves of π . The classes L and E_i generate the Néron-Severi group of X . The effective cone of X is generated by the 16 (-1) -curves on X , namely, $E_i, L - E_i - E_j$ for $i \neq j$, and $E = 2L - E_1 - E_2 - E_3 - E_4 - E_5$. The ample cone is dual to the effective cone, and is generated by the following 26 divisors:

$$L, L_i = L - E_i, L_{ij} = E + E_i + E_j, C_i = E + E_i, B_i = E + L_i$$

Let us adopt the convention that the notation L_{ij} implies that $i < j$. We have the following table of intersection numbers:

	L	L_i	L_{ij}	C_i	B_i
L	1	1	2	2	3
L_k	1	$1 - \delta_{ik}$	$1 + \delta_{ik} + \delta_{jk}$	$1 + \delta_{ik}$	$2 - \delta_{ik}$
$L_{k\ell}$	2	$1 + \delta_{ik} + \delta_{i\ell}$	$3 - \delta_{ik} - \delta_{j\ell}$	$2 - \delta_{ik} - \delta_{i\ell}$	$2 + \delta_{ik} + \delta_{i\ell}$
C_k	2	$1 + \delta_{ik}$	$2 - \delta_{ik} - \delta_{jk}$	$1 - \delta_{ik}$	$1 + \delta_{ik}$
B_k	3	$2 - \delta_{ik}$	$2 + \delta_{ik} + \delta_{jk}$	$1 + \delta_{ik}$	$2 - \delta_{ik}$

Let P be any point on X . We first assume that P does not lie on any of the 16 (-1) -curves on X . In that case, we will show that for any ample divisor D , a sequence of best D -approximation to P lies along a curve in the class $|C_i|$ for some i , or in the class $|L_i|$ for some i .

To do this, we introduce 10 subcones of the nef cone of X . For i between 1 and 5, we define the cone M_i to be the cone of ample divisors for which the divisor L_i has minimal degree amongst the classes L_j and C_j , and we define the cone N_i to be the cone of ample divisors for which the divisor C_i has minimal degree amongst the classes L_j and C_j . We will show that for every divisor in M_i , a sequence of best approximation to P lies along a curve in $|L_i|$, and similarly that for any divisor in N_i , a sequence of best approximation to P lies along a curve in $|C_i|$.

We first consider M_i . For simplicity, we will consider only M_1 ; the other four cases differ from this one only by a permutation of the indices. The cone M_1 has 26 generators:

$$L, L_1, L_1 + L_i, L_1 + C_i, L_{ij}, B_1, L + C_i, L_1 + C_1, B_1 + L_i$$

where in all cases we assume that $i \neq 1$ and $j \neq 1$. Of these generators, only L_1 is not big. Let F be the unique curve through P satisfying $F \in |L_1|$. Since we have assumed that P does not lie on a (-1) -curve, it follows that if a curve through P has intersection zero with one of these generators, then the curve in question must be F .

It suffices to show that for any generator of M_1 , a sequence of best approximation to P lies on F . For $L, L_1, L_1 + L_i, L_1 + C_i, L_{ij}$, and B_1 , this follows from Theorem 2.6 because F has degree one or zero.

In all other cases, F has degree two, and we can write them as follows:

$$\begin{aligned}
L + C_2 &= L_3 + L_{23} \\
L + C_3 &= L_2 + L_{23} \\
L + C_4 &= L_2 + L_{24} \\
L + C_5 &= L_2 + L_{25} \\
L_1 + C_1 &= L_2 + C_2 \\
B_1 + L_2 &= C_3 + L_{45} \\
B_1 + L_3 &= C_2 + L_{45} \\
B_1 + L_4 &= C_2 + L_{35} \\
B_1 + L_5 &= C_2 + L_{34}
\end{aligned}$$

For each generator D in the above list, we have written $D = D_1 + D_2$, where $D_i.F = 1$. Thus, by Theorem 3.1, F contains a sequence of best D -approximation to P unless P lies on some curve of smaller D -degree. Since we have assumed that P does not lie on a (-1) -curve, and since D lies in M_1 , it follows that P cannot lie on any curve of smaller D -degree than L_1 . (Note that the only candidates for a curve of smaller degree than two are those which intersect one of D_1 or D_2 trivially, and such curves are either (-1) -curves or else of the form L_i or C_i for some i .) We conclude that a sequence of best D -approximation to P lies on F .

Now to N_i . Again, we will consider only N_1 , the cone of divisors with respect to which C_1 has minimal degree amongst the L_j and C_j . Like M_1 , N_1 has 26 generators:

$$C_1, C_1 + L_i, C_1 + L_{1i}, L_{1i}, B_i, L + C_1, L_1 + C_1, C_1 + L_{ij}, B_2 + C_2$$

where again we assume that $i \neq 1$ and $j \neq 1$. Let F be the curve in $|C_1|$ containing P . As before, it suffices to show that for any generator of N_1 , a sequence of best approximation to P lies on F . For $C_1, C_1 + L_i, C_1 + L_{1i}, L_{1i}$, and B_i , this follows from Theorem 2.6 because F has degree one or zero, and all the generators contract no curves through

P other than F . For the remaining nine generators, we can write:

$$\begin{aligned}
L + C_1 &= L_2 + L_{12} \\
L_1 + C_1 &= L_2 + C_2 \\
B_2 + C_2 &= B_2 + C_2 \\
C_1 + L_{45} &= L_{15} + C_4 \\
C_1 + L_{35} &= L_{15} + C_3 \\
C_1 + L_{34} &= L_{14} + C_3 \\
C_1 + L_{25} &= L_{15} + C_2 \\
C_1 + L_{24} &= L_{14} + C_2 \\
C_1 + L_{23} &= L_{13} + C_2
\end{aligned}$$

In each case, we have written the generator D as $D = D_1 + D_2$, where $D_i.F = 1$. Thus, by Theorem 3.1, we conclude that since P does not lie on a (-1) -curve, a sequence of best D -approximation to P lies on F for any generator D of N_1 , and by Corollary 3.2 also for any D in N_1 .

It remains only to treat the case where P lies on a (-1) -curve. Since no three (-1) -curves meet at a point, P lies either on exactly one or exactly two (-1) -curves. Let's first assume that P lies on exactly one (-1) -curve. Since the automorphism group of X acts transitively on the set of (-1) -curves (see for example Theorem 2.1 of [Ho]), we may assume that this curve is E .

Consider the cone M_i . We will subdivide M_i into two subcones. The first is R_i , generated by the following twenty-two divisor classes:

$$L, L_i, L_i + L_j, L_{j\ell}, L_i + C_j, L_i + B_i, L + C_j, L + L_i + C_i$$

where j and ℓ are assumed to be different from i . The classes L , L_i , $L_i + L_j$, $L_{j\ell}$, $L + C_j$, and $L_i + B_i$ all either contract the component F_i of L_i through P , or else contract no curves through P and map F_i to a line. Thus, by Theorem 2.6 F_i is a curve of best approximation to P with respect to any of these divisor classes. For $L + C_j$, we see that $L.L_i = C_j.L_i = 1$, and so Theorem 3.1 implies that if F_i is not a curve of best $(L + C_j)$ -approximation to P , then a sequence of best approximation to P can be found on a curve contracted by either L or C_j . The only such curve through P is E , and we check that $E.(L + C_j) = 2$, so F_i is a curve of best $(L + C_j)$ -approximation to P (as is E). This only leaves $L + L_i + C_i$, which can be rewritten $L + L_i + C_i = L + L_j + C_j$ for any j . Since $L.L_i = L_j.L_i = C_j.L_i = 1$ for any $j \neq i$, we may apply Theorem 3.1 to argue that if F_i is not a curve of best $(L + L_i + C_i)$ -approximation to P , then by Theorem 3.1 a sequence

of best approximation to P must lie on some curve contracted by L , L_j , or C_j . The only such curves through P are E and the component of L_j through P . None of these have $(L + L_i + C_i)$ -degree less than 3 (which is the degree of F_i), so we conclude that F_i is a curve of best $(L + L_i + C_i)$ -approximation to P . Thus, by Corollary 3.2, it follows that F_i is a curve of best A -approximation to P for any A in R_i .

If A lies in M_i but not R_i , then it must lie in the cone R'_i , generated by:

$$L_{j\ell}, L_i + C_i, L_i + C_j, B_i, B_i + L_j, B_i + L_i, L + L_i + C_i, L + C_j$$

where j and ℓ range over every index different from i . All but the last two divisors in this list either contract E , or else contract no curves through E and map P to a line. For the last two, we see that E has the same degree as F_i , and we already know that F_i is a curve of best $(L + L_i + C_i)$ - and $(L + C_j)$ -approximation to P . Thus, we conclude that E is a curve of best A -approximation to P for every A in R'_i .

Consider the cone N_i . It is straightforward to check that except for $L + C_i$, every generator of N_i either contracts E or else contracts no curves through P and maps E to a line. We can rewrite $L + C_i = L_j + L_{ij}$ for any $j \neq i$, and note that $L_j.E = L_{ij}.E = 1$, so that by Theorem 3.1, E is a curve of best $(L + C_i)$ -approximation to P unless P lies on some curve of smaller $(L + C_i)$ -degree which is contracted by L or C_i . Since this is not the case, we conclude by Theorem 2.6 and Corollary 3.2 that E is a curve of best A -approximation to P for any A in N_i .

To finish the proof of Conjecture 2.7 for X , we just need to consider the case that P is the intersection point of two (-1) -curves. Up to automorphisms of X , we may assume that one of these (-1) -curves is E . Since the only (-1) -curves intersecting E are the E_i , we may assume that $P = E \cap E_i$ for some i .

Consider the cone J_1 , generated by the following divisor classes:

$$L + L_i, L_i, L_i + L_j, L_{j\ell}, L_i + C_j, L_i + B_i$$

where j and ℓ range over all values different from i . Each of these divisors either contracts the component F_i of L_i through P , or else contracts no curves through P and maps F_i to a line. By Theorem 2.6 and Corollary 3.2, we conclude that Conjecture 2.7 is true for any A in J_1 .

Next, consider the cone J_2 , generated by:

$$L, L_j, L + L_i, C_i, L_i + L_j, L_{ij}, L_{ji}, L_{j\ell}, L_j + C_\ell, L_{ij} + C_j, L_j + B_j$$

where j and ℓ range over all indices different from i and each other. Thus, the cone J_2 is generated by 41 divisor classes. Each of these

divisors either contracts E_i , or else contracts no curves through P and maps E_i to a line. By Theorem 2.6 and Corollary 3.2, we conclude that Conjecture 2.7 is true for all A in J_2 .

Finally, if A does not lie in J_1 or J_2 , then it must lie in J_3 , generated by:

$$L_i + B_i, C_i, L_{j\ell}, L_i + C_j, L_j + C_\ell, B_j, L_{ij} + C_j, L_j + B_j$$

where again j and ℓ are different from each other and i , for a total of 41 divisor classes in the list. Each class either contracts E , or else contracts no curves through P and maps E to a line. Thus, it follows from Theorem 2.6 and Corollary 3.2 that Conjecture 2.7 is true for any divisor A in J_3 .

This concludes the proof of Conjecture 2.7 for X . ♣

4. FURTHER REMARKS

These techniques will probably be able to prove Conjecture 2.7 for more split rational surfaces, but they will probably not suffice to prove the conjecture for a general rational surface. For example, let X be a blowup of \mathbb{P}^2 at six k -rational points in general position, embedded in \mathbb{P}^3 as a smooth cubic surface. The family \mathcal{F} of plane cubic curves passing through the set of blown up points in \mathbb{P}^2 is a three-dimensional (projective) linear subspace of the \mathbb{P}^9 of plane cubic curves. The Zariski closure Z of the set of cuspidal cubics is a closed subset of dimension 7. Therefore, $Z \cap \mathcal{F}$ has dimension at least one, so if the six points are chosen to lie on some cuspidal cubic whose cusp is not one of the six blown up points (which will generically be the case), there will be a one-dimensional family \mathcal{C} of cuspidal cubic curves on X .

Let P be the cusp of a cuspidal cubic C , and assume that P is k -rational. Then one can find a sequence on C which approximates P with constant of approximation $3/2$. In particular, no rational curve of minimal degree through P is a curve of best approximation to P , since that curve will generally be a conic (P cannot lie on any of the 27 lines on X). This suggests that any proof of Conjecture 2.7 for X will be beyond the techniques of this paper.

Note also that Vojta's Main Conjecture ([Vo], Conjecture 3.4.3) implies Conjecture 2.7 for many varieties. Vojta's Main Conjecture is as follows:

Conjecture 4.1 (Vojta's Main Conjecture). *Let X be a smooth algebraic variety defined over a number field k , with canonical divisor K . Let S be a finite set of places of k . Let L be a big divisor on X , and let D be a normal crossings divisor on X . Choose height functions h_K and h_L for K and L , respectively, and define a proximity function*

$m_S(D, P) = \sum_{v \in S} h_{D,v}(P)$ for D with respect to S , where $h_{D,v}$ is a local height function for D at v . Choose any $\epsilon > 0$. Then there exists a nonempty Zariski open set $U = U(\epsilon) \subset X$ such that for every k -rational point $P \in U(k)$, we have the following inequality:

$$(2) \quad M_S(D, P)H_K(P) \leq H_L(P)^\epsilon$$

(Note that our notation is multiplicative, rather than the additive notation used in [Vo]. Thus, Vojta's h is our $\log H$, and Vojta's m is our $\log M$.)

Assume that X is as in the conjecture, and has non-negative Kodaira dimension. Let P be any fixed k -rational point, lying on a rational curve $C \subset X$, and let D be any normal crossings divisor containing P . Since X has non-negative Kodaira dimension, there is a non-empty open set $U_1 \subset X$ such that there is a constant c satisfying $H_K(Q) > c > 1$ for all $Q \in U_1(k)$. Furthermore, if S is the set of archimedean places of k , we may write:

$$M_S(D, Q) \gg (\text{dist}(P, Q))^{-1}$$

where the implied constant is independent of Q . (If Q lies on the support of D , then $M_S(D, Q)$ is infinite, but this will not affect our proof.)

Let d be the constant of approximation for P on C . For any $\epsilon > 0$, let U_ϵ be the intersection of U_1 with the non-empty open set provided by Conjecture 4.1. Then for any point $Q \in U_\epsilon(k)$, Conjecture 4.1 implies:

$$\text{dist}(P, Q)H_L(P)^\epsilon > c$$

If we choose $\epsilon < d$, then it's clear that any sequence of best approximation to P must eventually lie in the complement of U_ϵ . If X is a surface, then this complement is a union of curves, in which case the truth of Conjecture 2.7 is clear. If X is of higher dimension, then a simple Noetherian induction shows that it suffices to assume that X contains no subvariety V whose Kodaira dimension is negative but whose dimension is at least two. Slightly more generally, we have therefore proven the following theorem:

Theorem 4.2. *Let V be a smooth algebraic variety with non-negative Kodaira dimension. Let $P \in V(k)$ be any rational point with a rational curve C/k on V through P , and assume that every subvariety of V with negative Kodaira dimension satisfies Conjecture 2.7 for P . Assume that Conjecture 4.1 is true for V . Then Conjecture 2.7 is true for P on V .*

One can also use our techniques to prove Conjecture 2.7 for certain special points on some varieties of nonnegative Kodaira dimension.

For example, let X be a smooth quartic in \mathbb{P}^3 containing a line L . This X is a K3 surface, and L is a smooth rational curve on X with self-intersection -2 . For a general such X , the effective cone of X is spanned by an elliptic curve E and L . (Indeed, one can choose E so that $E \cup L$ is a hyperplane section of X . See section 5 of [Ko] for details.)

The closure of the nef cone of X is therefore spanned by the divisors E and $D = 2E + 3L$. The former corresponds to a morphism from X to \mathbb{P}^1 giving X the structure of an elliptic surface, and D corresponds to the contraction of the -2 curve L . Let P be a point on L . Then clearly any sequence of best D -approximation to P lies on L , and by Theorem 2.1, a sequence of best $(E + L)$ -approximation to P also lies along L . Therefore, for any ample divisor A in the positive span of D and $E + L$, a sequence of best A -approximation to P must lie along L , by Theorem 3.2.

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