Curves arising from endomorphism rings of Kronecker modules

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Abstract

In this note, we prove that the endomorphism ring of a Kronecker module attached to a power series $\alpha \in k[[X]]$ is minimally generated by three generators, unless its degree d is less than 3. We prove this via the theory of algebraic curves, by proving that none of the affine curves arising from these endomorphism rings are planar for $d \geq 3$, but can always be embedded in $\mathbb{A}^3$.

1 Introduction

In their paper [3], Okoh and Zorzitto describe a family of k-algebras A_α attached to Kronecker modules P_α, where α is a power series with coefficients in an algebraically closed field k. The module P_α has a degree d, and Okoh and Zorzitto prove that if $d \leq 2$, then the algebra A_α is minimally generated as a k-algebra by d elements. They then ask how many elements will minimally generate an arbitrary algebra A_α. The purpose of this note is to show that the answer is $\min\{3, d\}$.

Okoh and Zorzitto prove their result for $d = 2$ using the techniques of algebraic curves. In this paper, we will take this philosophy further and derive general results using more sophisticated geometric tools. Precisely, in section 2, we recall the relevant definitions and notation from [3]. To keep the paper to a manageable length, we describe only the definitions and properties we use in this paper – for a full description of the Kronecker modules and associated constructions, we refer the reader to [3]. In section 3, we give a careful description of the geometry of the curves under consideration, and in section 4 we prove the main result. Section 4 in particular contains some quite technical algebraic geometry; we refer the reader to [1] for definitions and explanations of any unfamiliar geometric terms.

There are other questions posed by Okoh and Zorzitto in their paper, more closely tied to the geometry of the curves they construct. For example, they ask when the curves can be a complete intersection in some $\mathbb{A}^n$. We cannot answer this question in general, although if the curve is smooth, then the techniques of this paper, combined with a result of Serre-Murthy-Towber in [2], show that the algebra A_α corresponds to a complete intersection if and only if $d \leq 2$.

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2 Definitions and Notation

We review some of the notation and terminology of Kronecker modules from [3]. Let k be an algebraically closed field of arbitrary characteristic, and fix a linear functional $\alpha\colon k[X] \to k$. As described in [3], we may associate to this linear functional a deriver $\partial_\alpha\colon k[X] \to k[X]$ defined by:

$$\partial_\alpha(1) = 0, \text{ and } \partial_\alpha(X^j) = \sum_{i=0}^{j-1} \alpha(X^{j-i-1})X^i$$

Note in particular that if j is the smallest integer such that $\partial_\alpha(X^j) \neq 0$, then $\partial_\alpha(X^{j+1}) = \alpha(X^j)$ is a nonzero constant, and $\partial_\alpha(X^i) = 0$ for all $i \leq j$.

Okoh and Zorzitto also associate to α a Kronecker algebra P_α and its endomorphism ring:

$$A_\alpha = \mathrm{End}(P_\alpha)$$

which is the object with which this note is concerned. Since we will not use the precise definition of P_α, we refer the reader to [3] for its description.

We shall address the case in which P_α is indecomposable, in which case A_α is an integral domain. As noted in [3], it happens that P_α is indecomposable if and only if the following formal power series does not represent a rational function of X:

$$\sum_{n=0}^{\infty} \alpha(X^n)X^n$$

To summarize:

- The ring A_α is an integral domain.
- The functional α is irrational over $k[X]$ – in particular, α is not identically zero.

Explicitly, A_α is a subalgebra of the 2 by 2 matrix algebra $M_{22}(k(X))$ over $k(X)$. Okoh and Zorzitto define the generic matrix D associated to A_α:

$$D = \begin{bmatrix} p + \partial_\alpha(q) & -q \\ -\partial_\alpha(p + \partial_\alpha(q)) & \partial_\alpha(q) \end{bmatrix}$$

where p and q are a fixed pair of polynomials associated to α. (For details, see [3].) For each $i \geq 0$, $i \in \mathbb{Z}$, define:

$$\varphi_i = DX^i + \partial_\alpha(X^i)I$$

We then have:

$$A_\alpha = k[\varphi_0, \ldots, \varphi_{d-1}]$$

where $d = \deg \mathrm{Tr}D$.

By the Cayley-Hamilton theorem, D satisfies a quadratic equation over $k[X]$:

$$D^2 - (\mathrm{Tr}(D))D + \det(D) = 0 \tag{1}$$

Okoh and Zorzitto prove that the polynomial on the left side is irreducible (Lemma 4.9 of [3]). Thus, A_α is a domain, so that the fraction field $k(X, D)$ of A_α is a quadratic extension of the rational function field $k(X)$. Moreover, they also prove that $\deg \det(D) \leq \deg \mathrm{Tr}(D)$ (Proposition 3.11 of [3]). Finally, there is a distinguished maximal ideal J of A_α, defined by:

$$J = (\varphi_0, \varphi_1, \ldots) = (\varphi_0, \ldots, \varphi_{d-1})$$

3 Geometry

Consider $C_1 = \mathrm{Spec}\ A_\alpha$, the affine curve embedded in $\mathbb{A}_k^d$ via the coordinates $\varphi_0, \ldots, \varphi_{d-1}$. We also have the affine curve $C_2 = \mathrm{Spec}\ k[X, D] \subset \mathbb{A}_k^2$, which admits a birational map:

$$\psi \colon C_2 \to C_1$$

corresponding to the inclusion of rings $A_\alpha \hookrightarrow k[X, D]$. Our program will be to study the curve C_2, and then to deduce properties of C_1 via a study of the birational map ψ.

Note that the curve C_2 is an affine plane hyperelliptic curve, as given by the quadratic equation (1) (X and D are the coordinates). Let C be the projective closure of C_2 in $\mathbb{P}_k^2$. It is easy to see that there are exactly two points of C at infinity, and that these two points correspond to two points on the normalization of C. That is, C_2 is isomorphic to a projective hyperelliptic curve minus two points which are conjugate via the hyperelliptic involution. In particular, neither of these two points is a ramification point of the 2-to-1 map to $\mathbb{P}_k^1$.

We claim that C_1 is isomorphic to a (possibly singular) projective hyperelliptic curve minus one point which is not a ramification point of the hyperelliptic map. This claim will follow from the the following three intermediate claims:

1. The curve C_1 is not projective.

2. The map ψ is not surjective.

3. The map ψ is injective.

Once we have established these claims, we will know that C_1 is strictly bigger than C_2 but strictly smaller than a projective curve, and since C_2 is a projective curve minus two points, it will follow immediately that C_1 is a projective curve minus a single point. By the previous discussion, this single point will be one of the two points missing from C_2, which are not ramification points of the hyperelliptic map.

The first of the claims follows immediately from the fact that C_1 is affine. We will now proceed to prove the other two:

Lemma 3.1 *The map ψ is not an isomorphism.*

Proof. To prove this, we will construct a point (the only point, as it turns out) which is not in the image of ψ. This point Q_J corresponds to the maximal ideal J, generated by $\varphi_0, \ldots, \varphi_{d-1}$ (but recall also that J contains φ_i for all nonnegative i). To show that Q_J is not in the image of ψ, it suffices to show that J becomes the unit ideal upon the adjunction of X to A_α. This is easy: if i is the smallest positive integer such that $\partial_\alpha(X^i) \neq 0$, then $\partial_\alpha(X^i)$ must be constant, and hence

$$\varphi_i = DX^i + \gamma$$

for some nonzero $\gamma \in k$. Thus, $(k[X, D])J$ contains

$$(X^i)D - (DX^i + \gamma) = \gamma$$

and hence must be the unit ideal. ♣

Lemma 3.2 *The map ψ is injective.*

Proof. We will construct a birational inverse to ψ which is defined everywhere away from Q_J. In particular, for every point Q of $C_1 - \{Q_j\}$, we will construct a birational inverse of ψ which is defined at Q.

First, note that for each positive integer i, we have the identity:

$$\begin{aligned} \varphi_i &= DX^i + \partial_\alpha(X^i) \\ &= DX^i + X\partial_\alpha(X^{i-1}) + \alpha(X^{i-1}) \\ &= X\varphi_{i-1} + \alpha(X^{i-1}) \\ &= X\varphi_{i-1} + \alpha(X^{i-1}) \end{aligned}$$

We therefore obtain:

$$X = \frac{\varphi_i - \alpha(X^{i-1})}{\varphi_{i-1}}$$

Because $Q \neq Q_J$, we know that there exists some φ_i such that φ_i does not vanish at Q (that is, φ_i does not lie in the maximal ideal of A corresponding to Q). Thus, since $\alpha(X^{i-1}) \in k$, we have the following birational inverse of ψ which is defined at Q:

$$(\varphi_0, \frac{\varphi_i - \alpha(X^{i-1})}{\varphi_{i-1}})$$

and hence ϕ is injective, as desired. ♣

4 The canonical bundle of hyperelliptic curves

In this section, we will prove some geometric facts about hyperelliptic curves and their canonical bundles, from which it will follow that if $d \geq 3$, then C_1 cannot be embedded in $\mathbb{A}^2$. This section contains the most technically difficult algebraic geometry of the paper, and in particular we refer the reader to [1] for definitions and descriptions of the terms used in this section. We will, however, attempt to maintain a connection to the original algebraic nature of the problem, by explaining the algebraic significance of the geometric results, as they are proven.

The following definition is not standard, but is ideal for our purposes.

Definition 4.1 *Let C be a reduced and irreducible (but possibly singular) projective curve. Then we say that C is* hyperelliptic *if there is a finite* 2:1 *map $f : C \longrightarrow \mathbb{P}^1$.*

The projective closure of our curve C_1 is therefore hyperelliptic, in the sense of Definition 4.1. Our first step will be to show that C_1 is locally planar. Geometrically, this means that for every point P of C_1, there is an open neighborhood of P which is isomorphic to a planar curve. Algebraically, this means that for every maximal ideal M of A_α, there is a finite set $f_1, \ldots, f_r$ of elements $f_i \in A_\alpha - M$ such that the localization $A_\alpha[1/f_1, \ldots, 1/f_r]$ can be generated by two elements.

If the characteristic is different from two, this is a general fact about hyperelliptic curves:

Lemma 4.2 *Let C be a hyperelliptic curve (as above), with* 2:1 *map $f : C \longrightarrow \mathbb{P}^1$. Assume the characteristic of k is not 2. Then C is locally planar, and therefore a Gorenstein curve.*

Proof. The map f is finite by hypothesis, and flat by the usual criterion for flatness over a smooth curve, so the sheaf $E = f_*\mathcal{O}_C$ is a locally free rank two vector bundle on $\mathbb{P}^1$. The natural inclusion $\mathcal{O}_{\mathbb{P}^1} \hookrightarrow f_*\mathcal{O}_C$ followed by the trace map $\mathrm{Tr} : f_*\mathcal{O}_C \longrightarrow \mathcal{O}_{\mathbb{P}^1}$ is multiplication by 2, and therefore $E = \mathcal{O}_{\mathbb{P}^1} \oplus L$ for some line bundle L. (Here we use the fact that char $\neq 2$.)

If U is any open affine subset of $\mathbb{P}^1$, and y any local generator for $L|_U$ then (since E is an algebra bundle over $\mathbb{P}^1$) we can write $y^2 = u(x)y + v(x)$ for some functions $u(x)$, $v(x)$ on U. This shows that C is locally planar, hence Gorenstein and so has a dualizing line bundle ω_C. ♣

This shows that our curve C_1 from the previous section is locally planar in odd characteristic. In general, C_1 is always locally planar, since away from Q_J it is locally isomorphic to the planar curve C_2, and C_1 is smooth at Q_J, and hence locally planar. To prove this last claim, note that it is equivalent to show that the localization of A_α at J is a regular local ring. This is proven as follows:

Lemma 4.3 *The local ring $(A_\alpha)_J$ is a regular local ring. That is,* $\dim \overline{J}/\overline{J}^2 = \dim(A_\alpha) =$ 1*, where $\overline{J}$ is the maximal ideal of the local ring $(A_\alpha)_J$.*

Proof. Recall that $J = (\varphi_0, \ldots, \varphi_{d-1})$. Thus, for every i and j in the appropriate range, we have:

$$\varphi_i\varphi_j - \varphi_{i+1}\varphi_{j-1} \in \overline{J}^2$$

We may thus calculate:

$$\begin{aligned}
\varphi_i\varphi_j - \varphi_{i+1}\varphi_{j-1} &= (DX^i + \partial_\alpha X^i)(DX^j + \partial_\alpha X^j) \\
&\quad -((DX^{i+1} + \partial_\alpha X^{i+1})(DX^{j-1} + \partial_\alpha X^{j-1}) \\
&= D^2X^{i+j} + DX^j\partial_\alpha X^i + DX^i\partial_\alpha X^j + \partial_\alpha X^i\partial_\alpha X^j \\
&\quad -D^2X^{i+j} - DX^{j-1}\partial_\alpha X^{i+1} - DX^{i+1}\partial_\alpha X^{j-1} - \partial_\alpha X^{i+1}\partial_\alpha X^{j-1} \\
&= DX^j(\sum_{m=0}^{i-1} \alpha(X^m)X^{i-1-m}) + DX^i(\sum_{m=0}^{j-1} \alpha(X^m)X^{j-1-m}) \\
&\quad -DX^{i+1}(\sum_{m=0}^{j-2} \alpha(X^m)X^{j-2-m}) + DX^{j-1}(\sum_{m=0}^{i} \alpha(X^m)X^{i-m}) \\
&\quad +\partial_\alpha X^i\partial_\alpha X^j - \partial_\alpha X^{i+1}\partial_\alpha X^{j-1} \\
&= D(\sum_{m=0}^{i-1} \alpha(X^m)X^{i+j-1-m} + \sum_{m=0}^{j-1} \alpha(X^m)X^{i+j-1-m} \\
&\quad -\sum_{m=0}^{i} \alpha(X^m)X^{i+j-1-m} - \sum_{m=0}^{j-2} \alpha(X^m)X^{i+j-1-m}) \\
&\quad +\partial_\alpha X^i\partial_\alpha X^j - \partial_\alpha X^{i+1}\partial_\alpha X^{j-1} \\
&= D(\alpha(X^{j-1})X^i - \alpha(X^i)X^{j-1}) + \partial_\alpha X^i\partial_\alpha X^j - \partial_\alpha X^{i+1}\partial_\alpha X^{j-1} \\
&= \alpha(X^{j-1})\varphi_i - \alpha(X^i)\varphi_{j-1}
\end{aligned}$$

We use the convention that a sum from $m = 0$ to -1 is an empty sum. We therefore get:

$$\alpha(X^{j-1})\varphi_i - \alpha(X^i)\varphi_{j-1} \equiv 0 \bmod J^2$$

For some i, we have $\alpha(X^i) \neq 0$. This immediately implies that for all j:

$$\alpha(X^i)\varphi_j \equiv \alpha(X^j)\varphi_i \bmod J^2$$

Since the set $\{\varphi_i\}$ generates J as a vector space over k, it follows that J/J^2 is one-dimensional. Thus, $\overline{J}/\overline{J}^2$ is one-dimensional, as desired. ♣

Thus, our discussion will now focus on locally planar hyperelliptic curves. The strategy is to use the two-to-one map to $\mathbb{P}^1$ to link the algebra and geometry of C_1 to the algebra and geometry of $\mathbb{P}^1$. This is done via the dualizing sheaf ω_C on the projective curve C. We will first show that global sections of ω_C are invariant under the hyperelliptic involution on C.

Lemma 4.4 *Let C be a locally planar hyperelliptic curve (for example, the projective closure of C_1). Then the dualizing sheaf of C is $\omega_C = f^*\mathcal{O}_{\mathbb{P}^1}(g-1)$, where g is the arithmetic genus of g.*

Proof. Let g be the arithmetic genus of C, and set $w = f^*\mathcal{O}_{\mathbb{P}^1}(g-1)$. Then deg($w$)$= 2(g-1)$, and $h^0(C, w) \geq g$ since $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1)) = g$.

By Riemann-Roch,

$$h^0(w) - h^0(w^* \otimes \omega_C) = 2(g-1) + (1-g) = g-1,$$

so the condition that $h^0(w) \geq g$ implies that $h^0(w^* \otimes \omega_C) \geq 1$, hence $w = \omega_C$ since $w^* \otimes \omega_C$ is a line bundle of degree zero, and C is an irreducible curve. This proves the lemma ♣ .

As an aside, it is easy to describe what C looks like. Let $\widetilde{C}$ be the normalization of C. Then the composite map

$$\widetilde{C} \longrightarrow C \longrightarrow \mathbb{P}^1$$

is still 2:1, so we see that $\widetilde{C}$ is itself hyperelliptic, and that C is somehow constructed from $\widetilde{C}$ in such a way as to preserve the map to $\mathbb{P}^1$.

In fact, the singularities of C are as follows (with $u(x)$ and $v(x)$ as in the proof of Lemma 4.2):

- if $u^2(x) - 4v(x)$ has a zero of even order $2k$ at some point x of $\mathbb{P}^1$, then the singularity of C over x is obtained by taking the two conjugate points q, q' of $\widetilde{C}$ over x and gluing them together with contact order k.
- if $u^2(x) - 4v(x)$ has a zero of odd order $2k+1$ at some point x of $\mathbb{P}^1$, then the singularity of C over x is obtained by taking the ramification point q of $\widetilde{C}$ over x and "crimping" it, making it into the singularity

$$y^2 = x^{(2k+1)}$$

Even though C is singular, there is still an automorphism of C switching the sheets over $\mathbb{P}^1$, given locally (as an algebra automorphism of E) by $y \mapsto -u(x) - y$ (cf. the proof of Lemma 4.2).

Note that, since $h^0(\omega_C) = g$, all the global sections of ω_C on C are pulled back from the global sections of $\mathcal{O}_{\mathbb{P}^1}(g-1)$ on $\mathbb{P}^1$ and are therefore invariant under the hyperelliptic involution. This gives us the following:

Corollary 4.5 *If C is a hyperelliptic curve and $\sigma \in H^0(C, \omega_C)$ is a global section of the canonical bundle, then the order of vanishing of σ at a point p of C is the same as the order of vanishing of σ at p', the hyperelliptic conjugate of p. In particular, σ vanishes at p if and only if it vanishes at p'.*

We will now show that if C_1 can be embedded in $\mathbb{A}^2$, then we can find a global section of ω_C which does not vanish anywhere on C_1, but does have a zero on C. Since C_1 is not invariant under the hyperelliptic involution, this will provide our contradiction.

Lemma 4.6 *If C is a reduced and irreducible (but not necessarily smooth) projective curve of arithmetic genus g, and $p \in C$ is a smooth point of C such that $C - \{p\}$ can be embedded in $\mathbb{A}^2$, then C is a Gorenstein curve with dualizing line bundle $\omega_C = \mathcal{O}_C((2g-2) \cdot p)$.*

Proof. The curve C is clearly Gorenstein since $C - \{p\}$ is planar, and p is a smooth point. Let $i : C - \{p\} \hookrightarrow \mathbb{A}^2$ be the embedding. By the valuative criterion of properness, this extends to a map $\nu : C \longrightarrow \mathbb{P}^2$, with image curve C', differing from C only around $p' = \nu(d)$. If C' is a curve of degree e, then since p is the only point of C mapping to the line at infinity, we have $\nu^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_C(e \cdot p)$. In particular, $\nu^*\omega_{\mathbb{P}^2} = \nu^*\mathcal{O}_{\mathbb{P}^2}(-3) = \mathcal{O}_C(-3 \cdot p)$

By embedded resolution of singularities for curves on surfaces (valid in all characteristics) we can, by a sequence of blow-ups, resolve the singularity of C' at p' in $\mathbb{P}^2$. The result is a smooth surface S with proper map $\pi : S \longrightarrow \mathbb{P}^2$ and an embedding $j : C \hookrightarrow S$ making the following diagram commute:

$$\begin{array}{ccc} C & \overset{j}{\hookrightarrow} & S \\ & {\scriptstyle \nu}\searrow & \downarrow{\scriptstyle \pi} \\ & & \mathbb{P}^2 \end{array}$$

The exceptional divisors of the blowup meet C (in S) only at p. Since the canonical bundle of S is the pullback of the canonical bundle of $\mathbb{P}^2$ twisted by the exceptional divisors, we see that $j^*\omega_S$ is a multiple of p.

Before the blowups, the normal bundle of C' in $\mathbb{P}^2$ (when pulled back to C) was $\mathcal{O}_C(e^2 \cdot p)$. Again, since we blowup only at p' (or the corresponding point in the intermediate blowups) the normal bundle will change only at p, giving us that the normal bundle of C in S is again a multiple of p.

Finally, the adjunction formula $\omega_C = j^*\omega_S \otimes N_{C/S}$ then gives that the dualizing line bundle of C is a multiple of p. Since $\deg(\omega_C) = 2g-2$, this gives $\omega_C = \mathcal{O}_C((2g-2) \cdot p)$. ♣

Putting together Lemma 4.6 and Corollary 4.5, we arrive at the following result.

Corollary 4.7 *Let C be a hyperelliptic curve (in the sense of Definition 4.1), and $p \in C$ a smooth point which is not a fixed point of the hyperelliptic involution. If the arithmetic genus g of C is greater than 1, then there does not exist an embedding $C - \{p\} \hookrightarrow \mathbb{A}^2$.*

Proof. By Lemma 4.6, $\omega_C = \mathcal{O}_C((2g-2) \cdot p)$. If $g > 1$, then there is a global section of ω_C vanishing *only* at p. If p is not a fixed point of the hyperelliptic involution, then this contradicts Corollary 4.5 ♣.

The curve C_2 has arithmetic genus $g = d - 1$, and since C is C_2 plus two smooth points, it has the same genus as C_2. In particular, if $d \geq 3$, then $g > 1$. Thus, if $d \geq 3$, then the curve C_1 is not a planar curve, and hence the corresponding coordinate ring is not generated as a k-algebra by two elements. It is a well known geometric fact that every locally planar curve can be embedded in $\mathbb{A}^3$, so that A_α can always be generated by three elements. For the sake of completeness, the proof is included here:

Theorem 4.8 *Every locally planar affine curve can be embedded in $\mathbb{A}^3$.*

Proof. Let C be a locally planar affine curve, embedded in some affine space $C \subset \mathbb{A}^n$, where n is the minimal integer such that C embeds in $\mathbb{A}^n$. If $n \leq 3$, then we're done, so assume $n > 3$. View C as a subset of $\mathbb{P}^n$ via the embedding of $\mathbb{A}^n$ into $\mathbb{P}^n$ as a standard affine subset. Let $\pi \colon \mathbb{A}^n \to \mathbb{A}^{n-1}$ be any projection. Then π extends to a projection $\mathbb{P}^n \to \mathbb{P}^{n-1}$ away from a point P on the locus at infinity. Our strategy is to find a point P at infinity such that the map $\pi|_C$ is an isomorphism.

First, define the secant variety $S \subset \mathbb{P}^n$ to C. It is the projective closure of the union of all lines $L \subset \mathbb{P}^n$ such that $\#(L \cap C) \geq 2$. If $\pi|_C$ is to be one-to-one, we must choose P to lie outside of S.

In order to ensure that $\pi|_C$ is an isomorphism in a neighborhood of a point $Q \in C$, we need the line joining P to Q to intersect C transversely at Q. If Q is a smooth point of C, then this means that the line joining P to Q is not tangent to C at Q – in other words, we require that P does not lie on the tangent variety T of C, which is the closure of the union of all tangent lines to C.

If Q is a singular point of C, then since C is locally planar, there is a surface X_Q which is smooth at Q and contains C. For $\pi|_C$ to be an isomorphism near Q, it suffices to ensure that the line joining P to Q intersect X_Q transversely at Q. In other words, P must not lie in the tangent plane H_Q to X_Q at Q. Since C has only finitely many singular points, it follows that there are only finitely many such planes.

In summary, then, to ensure that $\pi|_C$ is an isomorphism, we need to choose P to lie outside the union $Y = S \cup T \cup H_{Q_1} \cup \ldots \cup H_{Q_r}$, where $Q_1, \ldots, Q_r$ are the singular points of C. Our goal is to show that Y does not contain all points at infinity.

First of all, each H_{Q_i} has dimension 2, and therefore if Y contains the $(n-1)$-dimensional hyperplane H_∞ at infinity, then $H_\infty \subset S \cup T$. (Recall that $n \geq 4$, so $n - 1 \geq 3$.) Similarly, since T is also (at most) two-dimensional ([1], proof of Proposition IV.3.5), we must have $H_\infty \subset S$.

If $H_\infty \subset S$, then if $L \subset H_\infty$ is a line through P, then L must intersect the projective closure $\overline{C}$ of C. Since C is not contained in H_∞, it follows that $\overline{C}$ intersects H_∞ in a finite set of points, and hence there are plenty of lines $L \subset H_\infty$ which contain P but are disjoint from $\overline{C}$. Thus, H_∞ is not contained in S, and therefore is not contained in Y either, so we can always choose P so that $\pi|_C$ is an isomorphism.

But this means that C can be embedded in $\mathbb{A}^{n-1}$, which contradicts the minimality of n. Therefore, n must be no greater than 3, as desired. ♣

We therefore summarize with the following theorem:

Theorem 4.9 *Let A_α be the endomorphism algebra associated to a Kronecker module of the sort described above. Then a minimal set of k-algebra generators has cardinality* $\min\{d, 3\}$.

Proof. We have already proved this for $d \geq 3$, and Okoh and Zorzitto deal with the case $d \leq 3$ in [3]. ♣

References

[1] [Har] Hartshorne, R., *Algebraic Geometry*, Springer Verlag, New York, 1977.

[2] [MT] Murthy, M.P. and Towber, J., “Algebraic vector bundles over $\mathbb{A}^3$ are trivial”, Invent. Math. 24 (1974), 173–189.

[3] [OZ] Okoh, F. and Zorzitto, F., “Curves arising from Kronecker modules”, to appear in Linear Algebra and its Applications, 2003.