An Arithmetic Analogue of Bezout's Theorem

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Abstract. In this paper, we prove two versions of an arithmetic analogue of Bezout's theorem, subject to some technical restrictions. The basic formula proven is $\deg(V)h(X\cap Y) = h(X)\deg(Y)+h(Y)\deg(X)+O(1)$, where X and Y are algebraic cycles varying in properly intersecting families on a regular subvariety $V_S \subset \mathbf{P}^N_S$. The theorem is inspired by the arithmetic Bezout inequality of Bost, Gillet, and Soulé, but improve upon it in two ways. First, we obtain an equality up to $O(1)$ as the intersecting cycles vary in projective families. Second, we generalise this result to intersections of divisors on any regular projective arithmetic variety.

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1. Introduction

In Arakelov theory, the height of a rational point is defined to be a certain intersection of arithmetic cycles. This suggests that theorems from classical intersection theory could be generalised to the arithmetic setting, where one might hope to prove arithmetic results.

The most famous theorem from classical intersection theory is Bezout's theorem, which relates the degree of the intersection of two algebraic cycles with the degrees of the original cycles. In their paper (Bost et al., 1994), Bost, Gillet, and Soulé prove several versions of an arithmetic analogue of Bezout's theorem. The basic idea of all of them is to bound from above the height of the cycle $X.Y$ in terms of the heights and degrees of two arithmetic cycles X and Y in $\mathbf{P}^N$. The purpose of this paper is to prove two different generalisations of this theorem.

Theorem 3.1 replaces the inequality of (Bost et al., 1994) with an equality up to $O(1)$, as X and Y vary in projective families satisfying two technical conditions (see Theorem 3.1). These are that each intersection of cycles in the two families should be proper, and that the family of such intersections should be flat. These conditions are quite necessary, as will be described below.

We consider intersections from a geometric point of view, rather than an arithmetic one. Given two curves X and Y on a surface V defined over $\mathbf{Q}$, the "height" of their intersection ought to be the height of the $\mathbf{Q}$-rational point(s) which lie on both curves. In other words, one should measure the height of $X_{\mathbf{Q}}.Y_{\mathbf{Q}}$, rather than the height of the arithmetic cycle $X.Y$ on some Arakelov model over $\mathbf{Z}$, which may include various finite components disjoint from the generic fibre.

For example, let $V = \mathbf{P}^2_{\mathbf{Q}}$, let X be the line $x = 5y$, and let Y be the line $x = 5z$. Considered over $\mathbf{Q}$, their intersection is the single point $[5\colon 1\colon 1]$, whose height is $\log 5$. If we consider the arithmetic intersection on the obvious model over $\mathbf{Z}$, the intersection cycle $X.Y$ is the sum of two irreducible components. The first is the closure of the point $[5\colon 1\colon 1]$ on the generic fibre, whose height is $\log 5$ (given a standard choice of Green form). The second is the entire line $x = 0$ on the fibre over 5, whose height will be the difference between the geometric intersection height and the arithmetic one. Geometrically, this extra component should be disregarded.

There is an obvious problem, however. The intersection of any two lines through the origin in $\mathbf{P}^2_{\mathbf{Q}}$ will be the origin itself. Lines through the origin can have arbitrary heights, so obviously there can be no exact formula in general for the height of the generic part of the intersection of X and Y in terms of the heights and degrees of X and Y.

We deal with this difficulty by allowing X and Y to vary in projective families, and allow equality only up to $O(1)$. As in the classical case, the primary obstacle to equality in the arithmetic Bezout theorem is improper intersections of X and Y. Provided that X and Y always intersect properly as they vary in their respective families, plus two other technical conditions, it is possible to replace the inequality of the arithmetic Bezout theorem from (Bost et al., 1994) with an equality up to $O(1)$. One can then further prove that in that case, the height of the intersection over $\mathbf{Q}$ and the arithmetic intersection over $\mathbf{Z}$ do not differ by more than $O(1)$ either, resulting in the geometric Theorem 3.1.

Theorem 3.1 also generalises of the Bezout theorem to intersections of divisors on more general arithmetic varieties. The theorem relies on a result on the continuity of a certain fibre integral, namely Theorem 2.1.

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2. Continuity of the Fibre Integral

Let M be a smooth projective complex variety of dimension d, and let T be a smooth quasi-projective complex variety of dimension e. Let

$p: M \times T \to T$ be projection onto the second factor, and for any closed point $t \in T$, let $i_t: M \to M \times T$ map x to (x, t). For a cycle Z on $M \times T$ which meets properly every fibre $M \times \{t\}$ of p, write $Z_t = i_t^* Z$. Similarly, if g is a Green form for Z, write $g_t = i_t^* g$ for the corresponding Green form for $p^{-1}(t)$, and if α is a continuous differential form on $M \times T$, write $\alpha_t = i_t^* \alpha$.

THEOREM 2.1. *Let Z_1 and Z_2 be two smooth cycles on $M \times T$, of respective codimensions p_1 and p_2, with $p_1 > 0$. Assume $Z_1.Z_2$ is also smooth. Let g be a Green form for Z_1 of log type along $|Z_1|$ and let α be a continuous closed (k,k)-form on $M \times T$, $k = d + e - p_1 - p_2$. Suppose that Z_1 and Z_2 meet properly and that, for any $t \in T$, Z_1, Z_2, and $|Z_1| \cap |Z_2|$ meet $M \times \{t\}$ properly, and consider the current $\alpha.g.\delta_{Z_2}$ on $M \times T$ and the currents $\alpha_t.g_t.\delta_{Z_{2,t}}$ on M. Then the integral*

$$\phi(t) = \int_M \alpha_t.g_t.\delta_{Z_{2,t}}$$

depends continuously on $t \in T$.

([Bost et al., 1994, Prop. 1.5.1] proves this and a bit more in the case dim$T$=1. There is no smoothness hypothesis on the Z_i in (Bost et al., 1994), and indeed it seems unlikely that the smoothness hypothesis should be necessary.)

Proof: The proof is a combination of techniques from (Bost et al., 1994) and Stoll, 1967. First, assume that $Z_2 = M \times T$, giving $\phi(t) = \int_M \alpha_t.g_t$. It suffices to prove the result for a single Green form g for Z_1 of log type along Z_1. If g' is another such form, then there exists $u \in A^{p_1-1,p_1-1}(M \times T)$ such that $g' - g - u$ is a ∂ and $\overline{\partial}$-closed form. This implies that $g'_t - g_t - u_t$ is also a ∂ and $\overline{\partial}$-closed form, so by Stokes' Theorem:

$$\int_M \alpha_t.g'_t = \int_M \alpha_t.g_t + \int_M \alpha_t.u_t$$

and the last integral is clearly a continuous function of t.

Now fix any $b \in T$. We will show that $\phi(t) = \int_M \alpha_t.g_t$ is continuous in an open neighbourhood of b. Fix a compact neighbourhood C of $b \in T$.

By blowing up Z_1, we obtain a smooth variety N and a proper map $\pi: N \to M \times T$. The exceptional divisor $E = \pi^{-1}(Z_1)$ is a divisor given locally by the equation $z = 0$, and π is an isomorphism on the complement of E. Moreover, since g is of log type along $|Z_1|$, there exists a ∂ and $\overline{\partial}$-closed form β and a smooth form γ locally on $\tilde{N} = \pi^{-1}(M \times C)$ such that locally on $\tilde{N} - E$, $\pi^*(g) = \beta \log |z|^2 + \gamma$.

We will use the following result of Stoll [Stoll, 1967, Theorem 4.9]:

LEMMA 2.2. *Let X be a pure m-dimensional, complex manifold. Let Y be a pure p-dimensional, normal complex space with $0 < p < m$. Define $q = m - p$. Let $f\colon X \to Y$ be a holomorphic map with pure q-dimensional fibres. Let κ be a non-negative integer. Let $g\colon X \to \mathbf{C}^s$ be a holomorphic map. Let $b \in Y$. Suppose that g is not identically zero on any branch of any fibre of f. Let χ be a continuous differential form of bidegree (q, q) on X. Let K be a compact subset of X such that $f^{-1}(b) \cap g^{-1}(0) \cap K \neq \emptyset$. Then:*

$$\int_{f^{-1}(w)\cap K} \nu_f (\log |g|)^\kappa \chi \to \int_{f^{-1}(b)\cap K} \nu_f (\log |g|)^\kappa \chi$$

as $w \to b$, where ν_f denotes the multiplicity of the fibre of f over w.

In fact, Stoll proves much more than this, but the above will suffice for our purposes. For details, see Stoll, 1967.

We apply Lemma 2.2 with $X = N$, $Y = T$, $\chi = \beta$ or γ, $g = z$, $\kappa = 0$ or 1, and $f = p \circ \pi$. The smoothness of Z_1, together with the fact that Z_1 intersects properly with each fibre of p, implies that the map f has pure dimensional fibres. The continuity of ϕ follows.

Now drop the assumption that $Z_2 = M \times T$. Let g_2 be a Green form of log type for Z_2. Then $g * g_2$ is a Green current for the intersection cycle $I = Z_1.Z_2$, and there is a Green form h of log type along I such that $h = g * g_2 + \partial u_1 + \overline{\partial} v_1$ for some currents u_1 and v_1.

Let $\omega = dd^c g + \delta_{Z_1}$. It is a smooth closed form of type (p_1, p_1), and we have:

$$g.\delta_{Z_2} = g * g_2 - \omega.g_2 = h - \omega.g_2 - \partial u_1 - \overline{\partial} v_1$$

We also have:

$$g_t.\delta_{Z_{2,t}} = h_t - \omega_t.g_{2,t} + \partial u_2 + \overline{\partial} v_2$$

for some currents u_2 and v_2. Since α_t is $\partial-$ and $\overline{\partial}-$closed, it follows that:

$$\int_M \alpha_t.g_t.\delta_{Z_{2,t}} = \int_M \alpha_t.h_t - \int_M \alpha_t.\omega_t.g_{2,t}$$

Thus, it is enough to prove Theorem 2.1 with (Z_1, Z_2, α, g) replaced by $(I, M \times T, \alpha, h)$ or $(Z_2, M \times T, \alpha\omega, g_2)$. This we have already done. ♣

3. The Main Theorem

Let S be the ring of integers of a number field K, and let V be a regular arithmetic subvariety of $\mathbf{P}^N_S$ of pure relative dimension n over S. Let

$Z_1 \subset V \times_S B_1$ and $Z_2 \subset V \times_S B_2$ be families of effective (d_1+1)- and (d_2+1)-cycles in V_S, parametrised by projective S-schemes B_1 and B_2 over S. Denote by p_i the map $p_i\colon Z_i \to B_i$. Let $\pi_i\colon V_S \times B_1 \times B_2 \to V_S \times B_i$ be the projection maps, and define $Z = \pi_1^*(Z_1).\pi_2^*(Z_2)$. Thus, Z is the family of cycles in V_S parametrised by $(B_1 \times B_2)(S)$, such that the cycle corresponding to (a, b) is the intersection of the cycles corresponding to a and b in Z_1 and Z_2, respectively.

THEOREM 3.1. *Assume that either $V = \mathbf{P}^N$, or that $d_1 = d_2 = n-1$. Asume further that the following conditions are satisfied:*

(1) $B_{i,K}$ is a smooth algebraic variety.

(2) The map $p_K\colon Z_K \to B_{1,K} \times B_{2,K}$ is flat, and the cycles $Z_{1,K}$, $Z_{2,K}$, and Z_K are smooth over the number field K.

(3) Any member of the family Z_1 intersects properly over S with any member of the family Z_2.

(4) Any two members of the family $Z_{i,K}$ are linearly equivalent on the generic fibre.

Let X and Y be K-rational members of the families Z_1 and Z_2, respectively. Assume that $d_1 + d_2 \geq N$. Then we have the following equality:

$$\deg_K(V)h(\overline{X_K.Y_K}) = \deg_K(X)h(Y) + \deg_K(Y)h(X) + O(1) \quad (1)$$

where the bound on the $O(1)$ term depends only on V_S, on the families Z_1 and Z_2, and on the degree $[K\colon \mathbf{Q}]$. (Note that the intersection above is on the generic fibre; $\overline{X_K.Y_K}$ denotes the closure in V_S of $X_K.Y_K$.)

Remarks: The regularity assumption in Theorem 3.1 is not always necessary. All of the degrees and heights used in the statement of Theorem 3.1 depend only on the geometric properties of the embedding of V in projective space, up to $O(1)$. Let L be the restriction of $\mathcal{O}(1)$ on $\mathbf{P}_S^N$ to V_S. If V_S is regular, then Theorem 3.1 will be true for any projective map $\phi\colon V_S \to \mathbf{P}_S^M$ such that $\phi_K^*(\mathcal{O}(1))_K \cong L_K$, regardless of the regularity of the image of ϕ.

Furthermore, note that if two purely horizontal divisors over S intersect properly on the generic fibre, then they must also intersect properly over S. Therefore, if Z_1 and Z_2 are families of divisors, we may replace criterion (3) with a criterion for proper intersections on the generic fibre, which is much easier criterion to verify.

Finally, note that condition (4) is automatically satisfied in the case that $V_S = \mathbf{P}_S^N$.

Proof: First, we will show that equation (1) holds if $h(\overline{X_K.Y_K})$ is replaced by $h(X.Y)$, the height of the intersection of X and Y over S.

In the case $V_S = \mathbf{P}^N_S$, this follows easily from [Bost et al., 1994, Prop 5.4.2] and Theorem 2.1. Fix an X and Y. By [Bost et al., 1994, Prop 5.4.2], we have the following equality:

$$\begin{aligned} h(X.Y) &= \deg_K(X)h(Y) + \deg_K(Y)h(X) \\ &\quad + b_{d_1 d_2}[K\colon \mathbf{Q}] \deg_K(X)\deg_K(Y) - \frac{1}{2}\int_{\mathbf{P}^N_{\mathbf{C}}} \delta_X g_Y \mu^{d_1+d_2-N+1} \end{aligned}$$

where $b_{d_1 d_2}$ is a constant depending only on d_1 and d_2, δ_X is the current of integration on $X_{\mathbf{C}}$, g_Y is a μ-normalised Green current for Y, of log type along the support of Y, and $\mu = c_1(\overline{\mathcal{O}(1)})$ is the Kähler form on $P^N_{\mathbf{C}}$. The crucial term is clearly the following complex integral:

$$\int_{\mathbf{P}^N_{\mathbf{C}}} \delta_X g_Y \mu^{d_1+d_2-N+1} \tag{2}$$

By Theorem 2.1, this integral varies continuously with X and Y. But X and Y are parametrised by the compact set $B_1^\sigma \times B_2^\sigma$, so this integral must be bounded independently of the choice of X and Y, and the result follows.

The case of intersection of divisors is more complicated. Let $\pi\colon V_S \to \mathrm{Spec}S$ be the structure morphism. For any arithmetic cycle Z, define $[Z]$ to be the well-defined element of $\widehat{CH}^*(V)$ given by associating to Z the normalised Green current g_Z satisfying $dd^c(g_Z) + \delta_Z = H(\delta_Z)$ and $H(g_Z) = 0$, where H is harmonic projection of currents.

LEMMA 3.2. *Let X be an effective divisor on V_K. As an abuse of notation, we will write $[X]$ for $[\overline{X}]$, where $\overline{X}$ denotes the closure of X in V_S. There exists a finite set $\mathcal{R}$ of vertical divisors in $\widehat{CH}^1(V)$ such that for any Y rationally equivalent to X on the generic fibre, one may write $[X] = [Y] + \pi^*\beta + R$ for some $\beta \in \widehat{CH}^1(S)$, and some $R \in \mathcal{R}$. Equality is taken in $\widehat{CH}^*(V)$.*

Proof of lemma: Fix a Y rationally equivalent to X on the generic fibre. Then $[X]-[Y]$ is linearly equivalent (in $\widehat{CH}^*(V)$) to a divisor D which is entirely supported on closed fibres of π.

Any component of D supported on an irreducible fibre of π must be a multiple of the entire fibre, and therefore the pullback of a divisor from $\mathrm{Spec}(S)$. All but finitely many fibres of π are irreducible, so we may choose $\beta \in \widehat{CH}^1(S)$ so that $R = [X]-[Y]-\pi^*\beta$ is supported only on reducible fibres of π. Moreover, since R is completely supported on finite fibres, its Green form is bounded. Up to linear equivalence, the Green form can be chosen to be harmonic and therefore constant, and so if β is chosen correctly the Green form (and hence the Green current) for R will be 0.

Let F be any reducible fibre of π. It is a closed subscheme of $\mathbf{P}_k^N$ for some finite field k, with divisors X_k and Y_k, induced from X and Y. The degrees of X_k and Y_k are bounded by the degree of X, so since k is finite, there are only finitely many possible choices for $X_k - Y_k$, independent of the choice of Y. Thus, up to linear equivalence, there are only finitely many choices for the components of R supported on F. Since there are only finitely many reducible fibres, there are only finitely many choices for R. ♣

Let $\hat{\mu}$ be the restriction of $\hat{c}_1(\overline{\mathcal{O}(1)})$ to $V_S \subset \mathbf{P}_S^N$. Fix fibres A_X of Z_1 and A_Y of Z_2, and write $[X] = [A_X] + \pi^*\beta_1 + R_1$ and $[Y] = [A_Y] + \pi^*\beta_2 + R_2$. Then we have:

$$\begin{aligned}
& \deg_K(V) h([X][Y]) \\
= \ & \deg_K(V)\widehat{\deg}(\pi_*(\hat{\mu}^{n-2}([A_X] + \pi^*\beta_1 + R_1)([A_Y] + \pi^*\beta_2 + R_2))) \\
= \ & \deg_K(V)\widehat{\deg}(\pi_*(\hat{\mu}^{n-2}[A_X][A_Y] + \hat{\mu}^{n-2}[A_X]R_2 + \hat{\mu}^{n-2}R_1[A_Y] \\
& + \hat{\mu}^{n-2}R_1R_2 + (\pi^*\beta_2)\hat{\mu}^{n-2}[A_X] + (\pi^*\beta_1)\hat{\mu}^{n-2}[A_Y])) \\
= \ & \deg_K(V)\widehat{\deg}(\beta_2\pi_*(\hat{\mu}^{n-2}[A_X]) + \beta_1\pi_*(\hat{\mu}^{n-2}[A_Y])) + O(1) \\
= \ & \deg_K(V)\deg_K(A_X)\widehat{\deg}(\beta_2) + \deg_K(V)\deg_K(A_Y)\widehat{\deg}(\beta_1) + O(1) \\
= \ & \deg_K(A_X)(h([Y]) - h([A_Y])) \\
& + \deg_K(A_Y)(h([X]) - h([A_X])) + O(1) \\
= \ & \deg_K([X])h([Y]) + \deg_K([Y])h([X]) + O(1)
\end{aligned}$$

where the $O(1)$ depends only on the R_i, $\hat{\mu}$, A_X, and A_Y, and hence only on V and the Z_i.

Unfortunately, $[X.Y]$ and $[X][Y]$ are not the same, since the star product of normalised Green forms is not itself normalised. Therefore, we must appeal to the following lemma to control the difference:

LEMMA 3.3. (Bost et al., 1994, Proposition 5.3.1). *We have the following equality in the group $\widehat{CH}^*(V)_{\mathbf{Q}}$:*

$$[X.Y] = [X][Y] - [0, H(g_X\delta_Y)]$$

where H denotes harmonic projection of currents, g_X is a normalised Green form for X, and δ_Y is the current of integration along Y.

Proof of lemma: See [Bost et al., 1994, Proposition 5.3.1]. ♣

Therefore, we get:

$$\deg_K(V)h(X.Y)$$

$$\begin{aligned}
&= \deg_K(V)h([X.Y]) \\
&= \deg_K(V)h([X][Y] - [0, H(g_X\delta_Y)]) \\
&= \deg_K([X])h([Y]) + \deg_K([Y])h([X]) + \deg_K(V)h([A_X][A_Y]) \\
&\quad -\widehat{\deg}(\pi_*[0, H(g_X\delta_Y)]) + O(1) \\
&= \deg_K(X)h(Y) + \deg_K(Y)h(X) - \int_{V(\mathbf{C})} g_X\delta_Y\mu^{n-2} + O(1)
\end{aligned}$$

As in the case of $V_S = \mathbf{P}_S^N$, Theorem 2.1 implies that this integral varies continuously with X and Y. But X and Y are parametrised by the compact set $B_1^\sigma \times B_2^\sigma$, so this integral must be bounded independently of the choice of X and Y, and the desired result follows.

We are not yet done with the proof of Theorem 3.1, since $X.Y$ may contain some components completely supported over finite points of Spec$S$, which will cause $h(X.Y)$ to be a bit greater than $h(\overline{X_K.Y_K})$. It remains only to show that these finite components are bounded in size over the families Z_i. Recall from the statement of the theorem that Z is defined to be the intersection cycle of Z_1 and Z_2.

LEMMA 3.4. *Let $Z_1 \subset V \times_S B_1$ and $Z_2 \subset V \times_S B_2$ be families of effective arithmetic cycles on a generically smooth, projective, regular arithmetic subvariety of $\mathbf{P}_S^N$ of pure dimension. Assume they satisfy conditions (2) and (3) listed in the statement of Theorem* 3.1. *Let X and Y be K-rational members of the families Z_1 and Z_2, respectively. Then $h(\overline{X_K.Y_K}) = h(X.Y) + O(1)$.*

Proof of lemma: Since p^σ is flat, it follows by the faithful flatness of $\mathbf{C}$ over K that the corresponding map $p_K\colon Z_K \to B_{1,K} \times B_{2,K}$ is also flat. Therefore, there is a non-empty open subset of Spec$S$ over which $p\colon Z \to B_1 \times B_2$ is flat, and hence only finitely many primes of S over which $(Z_1)_x$ and $(Z_2)_y$ can intersect improperly (and hence contribute finitely-supported components of $X.Y$).

The flatness of p_K also implies that for any prime π of S, the extension p_π of p_K to the completion K_π must also be flat; in particular, the fibres of p_π are of constant dimension r. Fix some finite prime π over which p is not flat.

CLAIM 3.5. *There exists a positive integer n, depending on π but not on X and Y, such that mod π^n, no intersection $X.Y$ contains the reduction of any cycle C on K_π of dimension greater than r.*

Proof of claim: Assume not. Then we may construct an infinite sequence of irreducible cycles C_n of fixed dimension $m > r$ such that for some X_n and Y_n, the intersection $X_n.Y_n$ contains the reduction of C_n modulo π^n. Since S/π^n is finite, we may further demand that $C_i \equiv C_j \bmod \pi^k$ for all i, j greater than or equal to k.

By the completeness of K_π, there exists a cycle C of dimension m such that $C \equiv C_n \bmod \pi^n$ for each n. Consider the two infinite sets $x_n = p_\pi(X_n)$ and $y_n = p_\pi(Y_n)$. By the properness of B_1 and B_2, these two sets must have accumulation points x and y, respectively. But then C has to be a subscheme of $p_\pi^{-1}(x).p_\pi^{-1}(y)$, and C has dimension $m > r$, which contradicts the flatness of p_π. ♣

The claim, together with the previous observation that the n in the claim can be taken to be 0 for all but finitely many π, implies that there is a uniform bound (independent of X, Y, and C) on the length of the module $\mathcal{O}_{X,C}/I(X \cap Y)$, where $I(X \cap Y)$ denotes the ideal of the scheme-theoretic intersection $X \cap Y$ in the local ring $\mathcal{O}_{X,C}$ of X along a component C of $X.Y$.

By [Fulton, 1998, Proposition 7.1 and the remarks in §20.1, p. 395], the intersection multiplicity of $X.Y$ along any proper component C is bounded above by the length of $\mathcal{O}_{X,C}/I(X \cap Y)$. Therefore, since by hypothesis X and Y intersect properly, it follows that for any prime π, the multiplicity of any component of $X.Y$ supported over π must be bounded independently of X and Y.

Moreover, there are only finitely many possibilities for any component of $X.Y$ supported over π, since the degree and dimension of any such component are bounded above independently of X and Y by $(\deg X)(\deg Y)$ and $\min\{\dim X, \dim Y\}$. Since there are only finitely many primes which can support any component of $X.Y$, $h(X.Y)$ and $h(\overline{X_K.Y_K})$ can only differ by $O(1)$. ♣

Hence Theorem 3.1 follows. ♣

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