V. Combinatorics and Lie Algebra.

REPRESENTATIONS OF THE LIE ALGEBRA $\mathfrak{sl}_2(\mathbb{C})$

Theorem 5.1. *Let $V = \bigoplus_{j\in\mathbb{Z}} V_j$ be a finite dimensional vector space over the complex numbers $\mathbb{C}$. Let $d_j := \dim_{\mathbb{C}} V_j$ for each $j \in \mathbb{Z}$ (so $d_j = 0$ except for finitely many indices). Assume that X and Y are linear transformations from V to V such that:*
(i) $\mathsf{X} : V_j \to V_{j+2}$ *for each* $j \in \mathbb{Z}$*;*
(ii) $\mathsf{Y} : V_j \to V_{j-2}$ *for each* $j \in \mathbb{Z}$*;*
(iii) *if V_j is nonzero then it is an eigenspace for $\mathsf{H} := \mathsf{XY} - \mathsf{YX}$ with eigenvalue j, for each $j \in \mathbb{Z}$.*
Then it follows that:
(a) $\mathsf{X} : V_j \to V_{j+2}$ *is injective for all* $j \leq -1$*;*
(b) $\mathsf{X} : V_j \to V_{j+2}$ *is surjective for all* $j \geq -1$*;*
(C) $\mathsf{X}^j : V_{-j} \to V_j$ *is bijective for all* $j \in \mathbb{N}$*.*
(d) *Consequently,*

$$d_{-2k} \leq d_{-2k+2} \leq \cdots \leq d_{-2} \leq d_0 \geq d_2 \geq \cdots \geq d_{2k-2} \geq d_{2k}$$

and

$$d_{-2k+1} \leq d_{-2k+3} \leq \cdots \leq d_{-1} = d_1 \geq \cdots \geq d_{2k-3} \geq d_{2k-1}$$

and $d_{-k} = d_k$ for all $k \in \mathbb{N}$.

Proof. Let V and $\mathsf{X}, \mathsf{Y}, \mathsf{H} : V \to V$ be as in the hypothesis. Since V is finite dimensional, there is a largest integer k such that V_k is nonzero. Let $\mathbf{v}_0$ be a nonzero vector in V_k; notice that $\mathsf{X}\mathbf{v}_0 = 0$ since $\mathsf{X}\mathbf{v}_0 \in V_{k+2} = 0$. For each $m \in \mathbb{N}$, let $\mathbf{v}_m := \mathsf{Y}^m\mathbf{v}_0$, and let U be the subspace of V spanned by $\{\mathbf{v}_0, \mathbf{v}_1, \ldots\}$. We claim that U is invariant under the linear transformations X, Y, and H.

Since $\mathsf{H} := \mathsf{XY} - \mathsf{YX}$, it suffices to prove that $\mathsf{X} : U \to U$ and $\mathsf{Y} : U \to U$. By construction of U, it is clear that Y maps U to U. To show that $\mathsf{X}(U) \subseteq U$ we show that there exist integer constants $c(k,m)$ such that $\mathsf{X}\mathbf{v}_m = c(k,m)\mathbf{v}_{m-1}$ for all $m \geq 1$. We prove this by induction on $m \geq 1$; for the basis $m = 1$ we have

$$\mathsf{X}\mathbf{v}_1 = \mathsf{XY}\mathbf{v}_0 = \mathsf{H}\mathbf{v}_0 + \mathsf{YX}\mathbf{v}_0 = k\mathbf{v}_0,$$

since $\mathbf{v}_0 \in V_k$ and $\mathsf{X}\mathbf{v}_0 = 0$, and so $c(k,1) = k$ is determined. Now assume that $c(k,m)$ exists, and apply induction and the fact that $\mathbf{v}_m \in V_{k-2m}$:

$$\mathsf{X}\mathbf{v}_{m+1} = \mathsf{XY}\mathbf{v}_m = \mathsf{H}\mathbf{v}_m + \mathsf{YX}\mathbf{v}_m = (k-2m)\mathbf{v}_m + c(k,m)\mathsf{Y}\mathbf{v}_{m-1} = (c(k,m) + k - 2m)\mathbf{v}_m,$$

from which we see that $c(k, m+1) = c(k,m) + k - 2m$, and hence U is invariant under the linear transformation X. Solving the recurrence for $c(k,m)$ with the initial condition $c(k,0) = 0$ results in $c(k,m) = km - 2\binom{m}{2} = (k-m+1)m$ for all $m \in \mathbb{N}$.

Now, since V is finite dimensional, there is a smallest integer ℓ such that $V_\ell \neq 0$. Since $\mathbf{v}_m \in V_{k-2m}$ for each $m \in \mathbb{N}$, this implies that $\mathbf{v}_m = 0$ for all sufficiently large m. Notice that if $c(k,m) \neq 0$ and $\mathbf{v}_{m-1} \neq 0$ then $\mathsf{X}\mathbf{v}_m = c(k,m)\mathbf{v}_{m-1} \neq 0$, so $\mathbf{v}_m \neq 0$; also, $c(k,m) = (k-m+1)m = 0$ if and only if $m = 0$ or $m = k+1$. Since $\mathbf{v}_0 \neq 0$ we see that $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k$ are nonzero. Also, if $\mathbf{v}_{k+1} \neq 0$ then $\mathbf{v}_m \neq 0$ for all $m \geq k+1$; since this is impossible, we must have $\mathbf{v}_{k+1} = 0$, and hence $\mathbf{v}_m = \mathsf{Y}^{m-k+1}\mathbf{v}_{k+1} = 0$ for all $m \geq k+1$. Since $\mathbf{v}_m \in V_{k-2m}$ for each $0 \leq m \leq k$, these vectors are linearly independent, and hence form a basis for U. Also, since $k \geq -k$ we see that $k \in \mathbb{N}$.

Now let $\mathbf{w}_1, \ldots, \mathbf{w}_t$ be vectors in V such that $(\mathbf{v}_0, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_t)$ is an ordered basis of V. We may choose such vectors $\mathbf{w}_i$ to be homogeneous (that is, each is contained in one of the subspaces V_j). With respect to this ordered basis, the linear transformations $\mathsf{X}, \mathsf{Y} : V \to V$ are represented by matrices of the following forms:

$$X = \begin{bmatrix} X' & 0 \\ * & \widetilde{X} \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} Y' & 0 \\ * & \widetilde{Y} \end{bmatrix}.$$

Here, X' and Y' represent X and Y acting on U, with respect to the ordered basis $(\mathbf{v}_0, \ldots, \mathbf{v}_k)$. Also, $\widetilde{X}$ and $\widetilde{Y}$ represent X and Y acting on the quotient space $V/U = \bigoplus_j V_j/U_j$ via the definitions

$$\widetilde{\mathsf{X}}(\mathbf{w} + U) := \mathsf{X}(\mathbf{w}) + U \quad \text{and} \quad \widetilde{\mathsf{Y}}(\mathbf{w} + U) := \mathsf{Y}(\mathbf{w}) + U$$

for all $\mathbf{w} \in V$. (Since $\mathsf{X}(U) \subseteq (U)$ and $\mathsf{Y}(U) \subseteq U$, these linear transformations $\widetilde{\mathsf{X}}$ and $\widetilde{\mathsf{Y}}$ are well-defined.) One easily checks that the hypotheses (i), (ii), and (iii) are satisfied by $\widetilde{\mathsf{X}}$ and $\widetilde{\mathsf{Y}}$ acting on V/U, and so we can apply induction on the dimension of V to assume that the theorem holds on V/U.

Now fix any $j \in \mathbb{Z}$, and consider the submatrix X_j of X with rows indexed by basis vectors of degree $j+2$ and columns indexed by basis vectors of degree j. This has the form

$$X_j = \begin{bmatrix} X'_j & 0 \\ * & \widetilde{X}_j \end{bmatrix}$$

in which X'_j and $\widetilde{X}_j$ are the corresponding submatrices of X' and of $\widetilde{X}$, respectively. By the structure of U determined above, X'_j is the empty matrix if $j \notin \{-k, -k+2, \ldots, k-2\}$, while if $j \in \{-k, -k+2, \ldots, k-2\}$ then it is the 1-by-1 matrix with sole entry $c(k, (k-j)/2)$, which is nonzero.

To prove (a), let $j \leq -1$, so that $j < k$. By induction, we know that $\widetilde{\mathsf{X}} : (V/U)_j \to (V/U)_{j+2}$ is injective. That is, the columns of $\widetilde{X}_j$ are linearly independent. Since $j < k$, if X'_j is not empty then its entry $c(k, (k-j)/2)$ is nonzero, while all other entries in the first row of X_j are zero. Hence, the columns of X_j are linearly independent. Therefore, $\mathsf{X} : V_j \to V_{j+2}$ is injective.

To prove (b), let $j \geq -1$, so that $j > -k-2$. By induction, we know that $\widetilde{\mathsf{X}} : (V/U)_j \to (V/U)_{j+2}$ is surjective. That is, the columns of $\widetilde{X}_j$ span V_{j+2}/U_{j+2}. Since $j > -k-2$, if

X'_j is not empty then $c(k,(k-j)/2) \neq 0$, while all other entries in the first row of X_j are zero. Hence, the columns of X_j span V_{j+2}.

To prove (c), let $j \in \mathbb{N}$ and consider the matrix X^j representing $\mathsf{X}^j : V \to V$ with respect to the ordered basis given above. Then

$$X^j = \begin{bmatrix} (X')^j & 0 \\ * & \widetilde{X}^j \end{bmatrix}$$

By induction on the dimension of V, $\widetilde{\mathsf{X}}^j : (V/U)_{-j} \to (V/U)_j$ is an isomorphism, for all $j \in \mathbb{N}$, and from the structure of U we see that $(\mathsf{X}')^j : U_{-j} \to U_j$ is an isomorphism for all $j \equiv k \pmod 2$ with $0 \leq j \leq k$, and hence for all $j \in \mathbb{N}$. Therefore $\mathsf{X} : V_{-j} \to V_j$ is an isomorphism for all $j \in \mathbb{N}$.

Part (d) follows immediately from parts (a), (b), and (c). □

There are a few remarks to be made about this theorem. First, for the induction step, in part (a) if $j = k$ then $\mathsf{X} : U_k \to U_{k+2}$ is not injective, since $U_k = \mathbb{C}\mathbf{v}_0$ while $U_{k+2} = 0$; similarly, in part (b) if $j = -k-2$ then $\mathsf{X} : U_{-k-2} \to U_{-k}$ is not surjective, since $U_{-k-2} = 0$ while $U_{-k} = \mathbb{C}\mathbf{v}_k$. The conditions $j \leq -1$ in part (a) and $j \geq -1$ in part (b) exclude these possibilities for all $k \in \mathbb{N}$.

Second, the conclusions (a), (b), and (c) of Theorem 1 have a dual form.

Proposition 5.2. *Under the hypotheses of Theorem 1:*
(a') $\mathsf{Y} : V_j \to V_{j-2}$ *is injective for all* $j \geq 1$*;*
(b') $\mathsf{Y} : V_j \to V_{j-2}$ *is surjective for all* $j \leq 1$*;*
(c') $\mathsf{Y}^j : V_j \to V_{-j}$ *is bijective for all* $j \in \mathbb{N}$*.*

Proof. In the proof of Theorem 1, just verify (a'), (b'), and (c') inductively at the appropriate point. □

Third, it is possible to find an ordered basis $(\mathbf{v}_0, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_t)$ as in the proof of Theorem 1 such that the matrices X and Y are block diagonal:

$$X = \begin{bmatrix} X' & 0 \\ 0 & \widetilde{X} \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} Y' & 0 \\ 0 & \widetilde{Y} \end{bmatrix}.$$

Then $W := \mathrm{span}_{\mathbb{C}}\{\mathbf{w}_1, \ldots, \mathbf{w}_t\}$ is an invariant subspace of V such that $V = U \bigoplus W$; that is, an *invariant subspace complementary to* U. This leads to a more detailed structure theorem than the one given above. However, the theorem above suffices for the applications I have in mind, and avoids the rather intricate proof of existence of complementary invariant subspaces.

A LITTLE LIE ALGEBRA

Fourth, what is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ anyway? Abstractly, a *complex Lie algebra* is a finite dimensional complex vectorspace L together with a bilinear operation

$[\cdot,\cdot] : L \times L \to L$ which is *skew-symmetric* ($[a, b] = -[b, a]$ for all $a, b \in L$) and satisfies the *Jacobi condition*:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \text{ for all } a, b, c \in L.$$

(Notice that $[a, a] = 0$ for all $a \in L$, as a consequence of skew-symmetry.) This bilinear operation is the *Lie bracket* on L. Lecture 8 of [W. Fulton and J. Harris, "Representation Theory: a First Course," GTM 129, Springer-Verlag, New York, 1991] gives a good explanation of the motivation for this definition. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has a three-element basis $\{x, y, h\}$ and the Lie bracket is determined by

$$[x, y] = h \quad\text{and}\quad [h, x] = 2x \quad\text{and}\quad [h, y] = -2y.$$

Exercise: Check that the Jacobi condition is satisfied in $\mathfrak{sl}_2(\mathbb{C})$.

A *representation* of the Lie algebra L is a finite dimensional nontrivial vectorspace $V \neq \{0\}$ and a linear transformation $T : L \to \mathrm{End}(V)$ such that $T([a, b]) = T(a)T(b) - T(b)T(a)$ for all $a, b \in L$. Here, $\mathrm{End}(V)$ denotes the algebra of all *endomorphisms* of V; that is, all linear transformations from V to V. The hypotheses of Theorem 1 imply that $T(x) := \mathsf{X}$ and $T(y) := \mathsf{Y}$ and $T(h) := \mathsf{H}$ determines a representation $T : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(V)$ on the vectorspace V. (The usual, but imprecise, language is to refer to V itself as a representation of the Lie algebra, the transformation T being understood.) For our purposes, the phrase "representation of $\mathfrak{sl}_2(\mathbb{C})$" can be taken as shorthand for "the hypotheses of Theorem 1 are satisfied".

Let V be a representation of a Lie algebra L. A subspace U of V is said to be *L-invariant* (or just "invariant") provided that $T(a)(U) \subseteq U$ for all $a \in L$. The representation V is *irreducible* if the only invariant subspaces of V are the trivial ones: $\{0\}$ and V itself. For each $k \in \mathbb{N}$, let $U(k)$ be the $(k+1)$-dimensional vectorspace with basis $\{\mathbf{v}_0, \ldots, \mathbf{v}_k\}$, with linear transformations $\mathsf{X}, \mathsf{Y} : U(k) \to U(k)$ defined by $\mathsf{Y}(\mathbf{v}_m) := \mathbf{v}_{m+1}$ for $0 \leq m < k$ and $\mathsf{Y}(\mathbf{v}_k) = 0$, and $\mathsf{X}(\mathbf{v}_m) = (k - m + 1)m\mathbf{v}_{m-1}$ for all $0 < m \leq k$ and $\mathsf{X}(\mathbf{v}_0) = 0$. These are exactly the invariant subspaces U which appear in the proof of Theorem 1.

Exercise: Show that $U(k)$ is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, for each $k \in \mathbb{N}$.

Exercise: Show that $U(k)$ for $k \in \mathbb{N}$ are *all* the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

The existence of a complementary invariant subspace, as in the third point above, implies that any representation of $\mathfrak{sl}_2(\mathbb{C})$ can be decomposed into a direct sum of irreducible representations $V = U(k_1) \oplus \cdots \oplus U(k_r)$. Moreover, in such a decomposition the multiset of indices $k_1, \ldots, k_r$ is determined by V, and thus does not depend on the particular decomposition. The general statement is that every representation of a semisimple Lie algebra can be decomposed as a direct sum of irreducible representations. This is known as Weyl's Theorem; a proof is given in Section 6.3 of [J.E. Humphreys, "Introduction to Lie Algebras and Representation Theory," GTM 9, Springer-Verlag, New York, 1972]. In particular, we have a nice structure theorem for representations of $\mathfrak{sl}_2(\mathbb{C})$; any such representation V can be decomposed, essentially uniquely, in the form

$$V = \bigoplus_{k=0}^{\infty} U(k)^{\oplus m(k)}$$

for some sequence of multiplicities $m : \mathbb{N} \to \mathbb{N}$ such that $m(0) + m(1) + \cdots$ is finite.

A nice way to see the uniqueness of the multiplicities $m(k)$ of the irreducible representations $U(k)$ in the above decomposition is by means of the *Poincaré polynomial* of the graded vector space $V = \bigoplus_{j \in \mathbb{Z}} V_j$; this is defined to be the Laurent polynomial

$$P(V; t) := \sum_{j=-\infty}^{\infty} (\dim_{\mathbb{C}} V_j) t^j.$$

For example, for each $k \in \mathbb{N}$, the Poincaré polynomial of $U(k)$ is

$$P(U(k); t) = t^{-k} + t^{-k+2} + \cdots + t^{k-2} + t^k = \frac{t^{-k-1} - t^{k+1}}{t^{-1} - t}.$$

Now, if V is decomposed as above, then on the level of Poincaré polynomials

$$P(V; t) = \sum_{k=0}^{\infty} m(k) \left(\frac{t^{-k-1} - t^{k+1}}{t^{-1} - t} \right).$$

Thus, if k is the largest exponent of t appearing in $P(V; t)$ then the multiplicity of $U(k)$ in V is $m(k) = \dim_{\mathbb{C}} V_k$; by induction, all of the remaining multiplicities $m(k-1), m(k-2), \ldots, m(1), m(0)$ can then be determined.
Exercise: If V and W are representations of $\mathfrak{sl}_2(\mathbb{C})$ such that $P(V; t) = P(W; t)$ then there is an isomorphism of vectorspaces $\phi : V \to W$ such that $\mathsf{X}\phi = \phi\mathsf{X}$ and $\mathsf{Y}\phi = \phi\mathsf{Y}$.

EXAMPLE: THE BOOLEAN REPRESENTATIONS OF $\mathfrak{sl}_2(\mathbb{C})$

Let A be a finite set, let $\mathcal{P}(A)$ be the set of all subsets of A, and let $\mathbb{C}\mathcal{P}(A)$ denote the complex vectorspace consisting of all formal linear combinations of subsets of A, so $\mathcal{P}(A)$ is a basis for $\mathbb{C}\mathcal{P}(A)$. Define two linear transformations X and Y on $\mathbb{C}\mathcal{P}(A)$ as follows. For any subset $S \subseteq A$,

$$\mathsf{X}(S) := \sum_{a \in A \smallsetminus S} (S \cup \{a\}) \quad \text{and} \quad \mathsf{Y}(S) := \sum_{a \in S} (S \smallsetminus \{a\}),$$

and these actions are extended linearly to all of $\mathbb{C}\mathcal{P}(A)$. Next, we verify that the hypotheses of Theorem 1 are satisfied.

Let $n := \#A$, let $0 \leq j \leq n$, let $\mathcal{P}_j(A)$ denote the set of j-element subsets of A, and let $S \in \mathcal{P}_j(A)$. Define a relation $S \sim T$ on $\mathcal{P}(A)$ to mean that $\#S = \#T$ and $\#(S \triangle T) = 2$, in which $\triangle$ denotes the symmetric difference of sets. One checks that

$$\mathsf{XY}(S) = j \cdot S + \sum_{T \sim S} T$$

and
$$\mathsf{YX}(S) = (n-j)\cdot S + \sum_{T\sim S} T,$$
from which it follows that S is an eigenvector of $\mathsf{H} := \mathsf{XY}-\mathsf{YX}$ with eigenvalue $j-(n-j) = 2j-n$. Thus, for each $0 \leq j \leq n$, $\mathbb{C}\mathcal{P}_j(A)$ is an eigenspace for H with eigenvalue $2j-n$, and from the definition of X and Y it is clear that $\mathsf{X} : \mathbb{C}\mathcal{P}_j(A) \to \mathbb{C}\mathcal{P}_{j+1}(A)$ and $\mathsf{Y} : \mathbb{C}\mathcal{P}_j(A) \to \mathbb{C}\mathcal{P}_{j-1}(A)$. This verifies conditions (i), (ii), and (iii) of the hypothesis of Theorem 1. Therefore, $\mathsf{X} : \mathbb{C}\mathcal{P}_j(A) \to \mathbb{C}\mathcal{P}_{j+1}(A)$ is injective if $j < n/2$ and surjective if $j \geq n/2$, and $\mathsf{X}^{n-2j} : \mathbb{C}\mathcal{P}_j(A) \to \mathbb{C}\mathcal{P}_{n-j}(\mathcal{A})$ is bijective if $j < n/2$. Part (d) of Theorem 1 implies some inequalities for binomial coefficients. There are much easier ways to prove these inequalities; however, we will make genuine use of these *Boolean representations* of $\mathfrak{sl}_2(\mathbb{C})$ in later sections.

By the binomial theorem, we see that the Poincaré polynomial of the Boolean representation $\mathbb{C}\mathcal{P}(A)$ is
$$P(\mathbb{C}\mathcal{P}(A); t) = (t^{-1}+t)^{\#A}.$$
We'll see a structural interpretation of this formula in the next section.

Exercise: Determine the multiplicities of the irreducible representations in the decompositions of the Boolean representations.

ASIDE: TENSOR PRODUCTS OF REPRESENTATIONS

Let V and W be representations of $\mathfrak{sl}_2(\mathbb{C})$, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_t\}$ be bases for V and W, respectively. The *tensor product* of V and W is the complex vectorspace $V \otimes W$ with basis $\{\mathbf{v}_i \otimes \mathbf{w}_j : 1 \leq i \leq d \text{ and } 1 \leq j \leq t\}$. Clearly, we have $\dim_{\mathbb{C}}(V \otimes W) = (\dim_{\mathbb{C}} V)(\dim_{\mathbb{C}} W)$. The tensor product of vectors satisfies
$$c(\mathbf{v} \otimes \mathbf{w}) = (c\mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (c\mathbf{w})$$
and
$$(\mathbf{u}+\mathbf{v}) \otimes \mathbf{w} = \mathbf{u} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}$$
for any scalar $c \in \mathbb{C}$ and vectors $\mathbf{u}, \mathbf{v} \in V$ and $\mathbf{w} \in W$. (It is linear in the W factor, as well.) Not every vector in $V \otimes W$ can be written in the form $\mathbf{v} \otimes \mathbf{w}$ with $\mathbf{v} \in V$ and $\mathbf{w} \in W$; these vectors are called *pure tensors*. Since $V \otimes W$ has a basis consisting of pure tensors, each vector in $V \otimes W$ is a linear combination of pure tensors. Now define linear transformations $\mathsf{X}, \mathsf{Y} : V \otimes W \to V \otimes W$ as follows:
$$\mathsf{X}(\mathbf{v} \otimes \mathbf{w}) := \mathsf{X}(\mathbf{v}) \otimes \mathbf{w} + \mathbf{v} \otimes \mathsf{X}(\mathbf{w})$$
and
$$\mathsf{Y}(\mathbf{v} \otimes \mathbf{w}) := \mathsf{Y}(\mathbf{v}) \otimes \mathbf{w} + \mathbf{v} \otimes \mathsf{Y}(\mathbf{w})$$
for any pure tensor $\mathbf{v} \otimes \mathbf{w}$ in $V \otimes W$, extended linearly to all of $V \otimes W$.

Exercise: Check that X and Y are well-defined linear transformations in $\mathrm{End}(V \otimes W)$.

Exercise: Check that X and Y defined above generate a representation of $\mathfrak{sl}_2(\mathbb{C})$ on $V \otimes W$.
Exercise: Check that the Poincaré polynomial of $V \otimes W$ is

$$P(V \otimes W; t) = P(V; t) \cdot P(W; t).$$

Since the Poincaré polynomial of $U(1)$ is $t^{-1} + t$, this shows that the Boolean representations of the previous section are exactly the tensor powers $U(1)^{\otimes n}$ of $U(1)$.
Exercise: Determine the multiplicities of the irreducible representations in $U(k) \otimes U(\ell)$ for all $k, \ell \in \mathbb{N}$.

GROUP ACTIONS AND EQUIVARIANT REPRESENTATIONS

Let V be a representation of $\mathfrak{sl}_2(\mathbb{C})$, and let G be a group of invertible linear transformations acting on V. We say that V is *G-equivariant* when $\mathsf{X}g = g\mathsf{X}$ and $\mathsf{Y}g = g\mathsf{Y}$ for all $g \in G$. Let

$$V^G := \{\mathbf{v} \in V : \ g(\mathbf{v}) = \mathbf{v} \text{ for all } g \in G\}$$

be the subspace of *G-invariant* vectors in V. (Notice that V^G really is a subspace of V.)

Proposition 5.3. *Let V be a representation of $\mathfrak{sl}_2(\mathbb{C})$ and let G be a group of invertible linear transformations on V. Assume that V is G-equivariant, and that the G-invariant subspace of V is not trivial: $V^G \neq \{0\}$. Then V^G is an $\mathfrak{sl}_2(\mathbb{C})$-invariant subspace of V, and hence is itself a representation of $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. We must check that $\mathsf{X}(V^G) \subseteq V^G$ and $\mathsf{Y}(V^G) \subseteq V^G$. So, let $\mathbf{v} \in V^G$ and let $g \in G$; then $g(\mathsf{X}\mathbf{v}) = \mathsf{X}g(\mathbf{v}) = \mathsf{X}\mathbf{v}$ and $g(\mathsf{Y}\mathbf{v}) = \mathsf{Y}g(\mathbf{v}) = \mathsf{Y}\mathbf{v}$. Since $\mathsf{X}\mathbf{v} \in V^G$ and $\mathsf{Y}\mathbf{v} \in V^G$ we have checked that $\mathsf{X}(V^G) \subseteq V^G$ and $\mathsf{Y}(V^G) \subseteq V^G$, so V^G is an invariant subspace of V. Since $V^G \neq \{0\}$, it is a representation of $\mathfrak{sl}_2(\mathbb{C})$ contained in V. □

For the combinatorial applications, we apply Proposition 3 to the Boolean representations. For a finite set A, each permutation $\sigma \in \mathcal{S}_A$ of the elements of A induces a permutation $\overline{\sigma} \in \mathcal{S}_{\mathcal{P}(A)}$ of the subsets of A:

$$\overline{\sigma}(S) := \{\sigma(a) : \ a \in S\}.$$

This permutation $\overline{\sigma}$ of the set $\mathcal{P}(A)$ of basis elements extends linearly to an invertible linear transformation on $\mathbb{C}\mathcal{P}(\mathcal{A})$, which we again denote by $\overline{\sigma}$. In this way, any permutation group $G \leq \mathcal{S}_A$ can be regarded as a group of invertible linear transformations on $\mathbb{C}\mathcal{P}(A)$.

Corollary 5.4. *Let A be a finite set, and let $G \leq \mathcal{S}_A$ be a group of permutations on A. Then $\mathbb{C}\mathcal{P}(A)^G$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. For each $0 \leq j \leq n := \#A$, let $\mathbf{v}_j$ be the sum of all j-element subsets of A. Then $\mathbf{v}_j$ is G-invariant for each $0 \leq j \leq n$, so $\mathbb{C}\mathcal{P}(A)^G \neq \{0\}$. By Proposition 3, to complete the proof it suffices to check that $\mathbb{C}\mathcal{P}(A)$ is G-equivariant. We will show that $\mathsf{X}\overline{\sigma} = \overline{\sigma}\mathsf{X}$ for all $\sigma \in G$; the analogous verification for Y is similar. To show that $\mathsf{X}\overline{\sigma} = \overline{\sigma}\mathsf{X}$ it suffices to verify this equation when applied to every vector in the basis $\mathcal{P}(A)$. So, consider any subset $S \subseteq A$, and calculate that

$$\mathsf{X}\overline{\sigma}(S) = \sum_{a \in A \smallsetminus \overline{\sigma}(S)} (\overline{\sigma}(S) \cup \{a\}) = \sum_{a \in A \smallsetminus S} \overline{\sigma}(S \cup \{a\}) = \overline{\sigma}\mathsf{X}(S).$$

Similarly, $\mathsf{Y}\overline{\sigma} = \overline{\sigma}\mathsf{Y}$ for every $\sigma \in G$; hence, $\mathbb{C}\mathcal{P}(A)$ is G-equivariant, completing the proof. □

Exercise: Let $G \leq \mathcal{S}_A$ be a group of permutations acting on the finite set A. Denote the orbits of G acting on $\mathcal{P}(A)$ by $\mathcal{O}_1, \cdots, \mathcal{O}_d$, and for $1 \leq i \leq d$ define $\mathbf{v}_i := \sum_{S \in \mathcal{O}_i} S$. Show that $\{\mathbf{v}_1, \cdots, \mathbf{v}_d\}$ is a basis for $\mathbb{C}\mathcal{P}(A)^G$.

APPLICATION: UNLABELLED GRAPHS AND EDGE-RECONSTRUCTION

An *unlabelled graph* is more properly described as an isomorphism class of graphs. Let $g(n, k)$ be the number of unlabelled simple graphs with n vertices and k edges.
Exercise: Verify the entries of the following table:

k	0	1	2	3	4	5	6	7	8	9	10
$g(5,k)$	1	1	2	3	5	6	5	3	2	1	1

This sequence is *palindromic* (it is equal to its reverse sequence) and *unimodal* (it is weakly increasing up to its maximum, and afterwards it is weakly decreasing). A palindromic unimodal sequence of positive integers is a "smoking gun" indicating the presence of a representation of $\mathfrak{sl}_2(\mathbb{C})$. By considering the Poincaré polynomials, we see that the decomposition of this representation into irreducibles is

$$U(10) \oplus U(6) \oplus U(4) \oplus U(2) \oplus U(2) \oplus U(0).$$

To construct the desired representations of $\mathfrak{sl}_2(\mathbb{C})$, let $N_n := \{1, 2, \ldots, n\}$ and let $A := \mathcal{B}(n, 2)$ be the set of 2-element subsets of N_n. A (labelled) simple graph with vertex-set N_n corresponds bijectively with a subset of $\mathcal{B}(n, 2)$, that is, with an element of $\mathcal{P}(A)$. The symmetric group $\mathcal{S}_n$ has a natural action as a group of permutations on $A = \mathcal{B}(n, 2)$; this induces a group of permutations on $\mathcal{P}(A)$. The orbits of $\mathcal{S}_n$ acting on $\mathcal{P}(A)$ are the isomorphism classes of all graphs with vertex-set N_n; these correspond bijectively with the isomorphism classes of all n-vertex graphs. By the exercise at the end of the previous section, the Poincaré polynomial of $\mathbb{C}\mathcal{P}(A)^{\mathcal{S}_n}$ is thus

$$P(\mathbb{C}\mathcal{P}(A)^{\mathcal{S}_n}; t) = \sum_{k=0}^{n(n-1)/2} g(n, k) t^{2k-n(n-1)/2}.$$

By Corollary 4, we know that $\mathbb{C}\mathcal{P}(A)^{\mathcal{S}_n}$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$, and Theorem 1(d) implies that the sequence $(g(n, k) : \ 0 \le k \le n(n-1)/2)$ is palindromic and unimodal. In summary, we have proved the following proposition.

Proposition 5.5. *For each $n \in \mathbb{N}$, the sequence $(g(n, k) : \ 0 \le k \le n(n-1)/2)$ is palindromic and unimodal.*

We can extract more information from Theorem 1 in this situation. To do this, define the *edge-deck* of an unlabelled graph G to be the multiset $D(G)$ of unlabelled graphs obtained by deleting each edge of G one at a time. Let $m(J, G)$ denote the multiplicity with which J appears in the edge-deck of G. Here is an example: For another example, consider the complete bipartite graph $K_{1,3}$ and the graph $K_3 \cup K_1$ consisting of a triangle and an isolated vertex. It is easy to check that $D(K_{1,3}) = D(K_3 \cup K_1)$, consisting of $K_{1,2} \cup K_1$ with multiplicity three. The *Edge-Reconstruction Conjecture* is that this is the only degeneracy in the transformation $G \mapsto D(G)$. More precisely, the conjecture is that

FIGURE 1

if G and H are unlabelled simple graphs such that $D(G) = D(H)$, then either $G = H$ or one of them is $K_{1,3}$ and the other is $K_3 \cup K_1$.

From the fact that $\mathbb{C}\mathcal{P}(\mathcal{B}(n,2))^{\mathcal{S}_n}$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$, we obtain the following result, due to Stanley, that the Edge-Reconstruction Conjecture holds for all "dense" graphs.

Theorem 5.6 (Stanley)**.** *For each $n \in \mathbb{N}$ and $n(n-1)/4 < k \leq n(n-1)/2$, the Edge-Reconstruction Conjecture holds for graphs with n vertices and k edges.*

Proof. The unlabelled simple graphs G with n vertices correspond bijectively with the orbits of $\mathcal{S}_n$ acting on $\mathcal{P}(\mathcal{B}(n,2))$. The size of the orbit $\mathcal{O}(G)$ corresponding to G is $n!/\#\mathrm{aut}(G)$, in which $\mathrm{aut}(G)$ is the automorphism group of G. We identify G with the sum of all (labelled) graphs $\gamma \subseteq \mathcal{B}(n,2)$ in the corresponding orbit $\mathcal{O}(G)$, so G is an $\mathcal{S}_n$-invariant vector in $V := \mathbb{C}\mathcal{P}(\mathcal{B}(n,2))$; these vectors form a basis for $V^{\mathcal{S}_n}$. It is convenient to fix an arbitrary labelled graph $\gamma_0 \in \mathcal{O}(G)$, and to observe that

$$G = \sum_{\gamma \in \mathcal{O}(G)} \gamma = \frac{1}{\#\mathrm{aut}(G)} \sum_{\sigma \in \mathcal{S}_n} \overline{\sigma}(\gamma_0).$$

Next, we consider the image of G under the transformation Y:

$$\begin{aligned} \mathsf{Y}(G) &= \sum_{\gamma \in \mathcal{O}(G)} \mathsf{Y}(\gamma) = \frac{1}{\#\mathrm{aut}(G)} \sum_{\sigma \in \mathcal{S}_n} \sum_{e \in \gamma_0} \overline{\sigma}(\gamma_0 \smallsetminus \{e\}) \\ &= \frac{1}{\#\mathrm{aut}(G)} \sum_{J \in D(G)} m(J,G)(\#\mathrm{aut}(J)) \cdot J. \end{aligned}$$

Accordingly, let $\Delta : V^{\mathcal{S}_n} \to V^{\mathcal{S}_n}$ be the endomorphism of $V^{\mathcal{S}_n}$ represented by the diagonal matrix with (G,G)-entry given by $\Delta_{G,G} := \#\mathrm{aut}(G)$ for every unlabelled graph G with n vertices. Let M be the endomorphism of $V^{\mathcal{S}_n}$ represented by the matrix with (J,G)-entry given by $M_{J,G} := m(J, G)$ for each pair of unlabelled n-vertex graphs. Notice that

$$M(G) = \sum_J m(J, G) \cdot J.$$

We have verified the equation $\mathsf{Y} = \Delta M \Delta^{-1}$. Proposition 2(a') implies that $\mathsf{Y} : V_j^{\mathcal{S}_n} \to V_{j-2}^{\mathcal{S}_n}$ is injective for $j \geq 1$. This is equivalent to the condition that $M : V_j^{\mathcal{S}_n} \to V_{j-2}^{\mathcal{S}_n}$ injective for $j \geq 1$. This implies that the transformation $G \mapsto D(G)$ from n-vertex unlabelled graphs to their edge-decks is injective provided that $n(n-1)/4 < k$, completing the proof. $\square$

APPLICATION: THE q-BINOMIAL COEFFICIENTS

First, we recall a few facts about the q-binomial coefficients; these are all proved in Section 5 of the C&O 330 Course Notes (in the reserve folder at the Library).

For nonnegative integers $a, b \in \mathbb{N}$, let $\mathcal{L}(a, b)$ be the set of all lattice paths in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to (a, b). It is easy to see that $\#\mathcal{L}(a, b) = \binom{a+b}{b}$ by constructing a bijection between $\mathcal{L}(a, b)$ and the set $\mathcal{B}(a + b, b)$ of b-element subsets of N_{a+b}. For P a lattice path in $\mathcal{L}(a, b)$, let area(P) denote the area of the compact subset of $\mathbb{R}^2$ bounded by the path P and the line segments from $(0, 0)$ to $(0, b)$, and from $(0, b)$ to (a, b). For a finite set $S \subseteq \mathbb{N}$ of nonnegative integers, let $\mathrm{sum}(S) := \sum_{s \in S} s$.

Exercise: Define a bijection between $\mathcal{L}(a, b)$ and $\mathcal{B}(a + b, b)$ such that if $P \in \mathcal{L}(a, b)$ corresponds to $S \in \mathcal{B}(a + b, b)$ then

$$\mathrm{area}(P) + \binom{b+1}{2} = \mathrm{sum}(S).$$

For present purposes, we will define the *q-binomial coefficients* to be the generating functions of these sets of lattice paths with respect to area:

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q := \sum_{P \in \mathcal{L}(a,b)} q^{\mathrm{area}(P)}.$$

From another point of view, this is the generating function for all partitions λ such that $\lambda_1 \leq a$ and $\ell(\lambda) \leq b$, with each such partition λ contributing $q^{|\lambda|}$. It is easy to see that the generating function for all subsets $S \subseteq N_n$ with S contributing $x^{\#S} q^{\mathrm{sum}(S)}$ is

$$(1 + qx)(1 + q^2 x) \cdots (1 + q^n x).$$

By using the above bijection we derive the q-Binomial Theorem: for any $n \in \mathbb{N}$,

$$(1 + qx)(1 + q^2 x) \cdots (1 + q^n x) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$

A slightly more intricate argument results in an explicit formula for the q-binomial coefficients. For each $m \in \mathbb{N}$, let

$$[m] := 1 + q + q^2 + \cdots + q^{m-1} = \frac{1 - q^m}{1 - q}$$

and let $[m]! := [m][m - 1] \cdots [3][2][1]$. Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n - k]!},$$

a lovely result. (For a proof of this we refer back to Section 5 of the C&O 330 Course Notes.) For example, since $[6] = [3](1 + q^3) = [3][2](1 - q + q^2)$, we calculate that

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix}_q = \frac{[7][6][5]}{[3][2][1]} = [7][5](1 - q + q^2)$$
$$= 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12}$$

From the combinatorial definition, it is clear that the q-binomial coefficients are polynomials in q with nonnegative integer coefficients. We observe in this example that the sequence of coefficients is palindromic and unimodal, and we suspect the existence of a representaation of $\mathfrak{sl}_2(\mathbb{C})$ which decomposes as

$$U(12) \oplus U(8) \oplus U(6) \oplus U(4) \oplus U(0).$$

To construct the desired representations of $\mathfrak{sl}_2(\mathbb{C})$, fix $a, b \in \mathbb{N}$ and let $A := N_b \times N_a$. Think of this as the Ferrers diagram of the partition consisting of b parts of size a. From the above remarks, we want the generating function for all partitions λ with $F_\lambda \subseteq A$, with respect to the number of boxes of F_λ. To apply Corollary 4, we consider the set $\mathcal{P}(A)$ of all subsets of A, and look for a group $G \leq \mathcal{S}_A$ of permutations on A such that each orbit of G acting on $\mathcal{P}(A)$ contains the Ferrers diagram of exactly one partition. Having found such a group, it follows that the dimension of $\mathbb{C}\mathcal{P}_j(A)^G$ is the number of partitions λ with $F_\lambda \subseteq A$ and $|\lambda| = j$. That is, in terms of the Poincaré polynomial of $\mathbb{C}\mathcal{P}(A)^G$,

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = q^{ab/2} P(\mathbb{C}\mathcal{P}(A)^G; q^{1/2}).$$

Since $\mathbb{C}\mathcal{P}(A)^G$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$ by Corollary 4, it follows from Theorem 1(d) that the sequence of coefficients of the q-binomial coefficient is palindromic and unimodal.

To complete the proof, we must find an appropriate group $G \leq \mathcal{S}_A$ of permutations acting on A. Consider an arbitrary subset B of the boxes of A. If we allow ourselves to arbitrarily permute the boxes within each column, then there is a unique subset B' in the orbit of B such that if a box (i, j) is in B' then every box above (i, j) is also in B'. Now, if we allow ourselves to permute the columns of A – but keeping the order of the boxes fixed within each column – then there is a unique subset B'' in the orbit of B' such that if a box (i, j) is in B'' then every box either above or to the left of (i, j) is also in B''. This means that B'' is the Ferrers diagram of a partition. Let G be the group generated by the permutations of these two types; this is the *wreath product* $\mathcal{S}_a[\mathcal{S}_b]$ of the symmetric groups $\mathcal{S}_a$ and $\mathcal{S}_b$. More explicitly, let $H := \mathcal{S}_b \times \cdots \times \mathcal{S}_b$ with a factors, in which the j-th factor is acting as permutations on column j of A. For each $\sigma \in \mathcal{S}_a$, define $\widetilde{\sigma} \in \mathcal{S}_A$ by $\widetilde{\sigma}(i, j) := (i, \sigma(j))$ for all $(i, j) \in A$. Then $K := \{\widetilde{\sigma} : \sigma \in \mathcal{S}_a\}$ is a subgroup of $\mathcal{S}_A$ isomorphic to $\mathcal{S}_a$. The wreath product $\mathcal{S}_a[\mathcal{S}_b]$ is generated by all permutations in $H \cup K$. (In fact, any permutation in $\mathcal{S}_a[\mathcal{S}_b]$ can be written as a composition $k \circ h$ with $k \in K$ and $h \in H$.) The following exercise completes the proof of Proposition 7.

Exercise: Show that for each subset $B \subseteq A$, there is a unique subset $B'' \subseteq A$ in the orbit of $\mathcal{S}_a[\mathcal{S}_b]$ acting on $\mathcal{P}(A)$ such that B'' is the Ferrers diagram of a partition.

In summary, we have proved the following proposition.

Proposition 5.7. *For any $a, b \in \mathbb{N}$, the coefficients of of* $\begin{bmatrix} a+b \\ b \end{bmatrix}_q$ *form a palindromic unimodal sequence of positive integers.*

LOGARITHMIC CONCAVITY

Turning back to a question from the first homework assignment, let $q(n,k)$ be the number of connected (labelled) graphs with vertex-set $\{1, 2, \ldots, n\}$ and with k edges. We were able to compute the polynomials $Q_n(t) := \sum_{k=0}^{n(n-1)/2} q(n,k)t^k$ quite efficiently. For example, $Q_6(t)$ is

$$1296t^5 + 3660t^6 + 5700t^7 + 6165t^8 + 4945t^9 + 2997t^{10} + 1365t^{11} + 455t^{12} + 105t^{13} + 15t^{14} + t^{15}.$$

The sequence of coefficients of $Q_6(t)$ is evidently unimodal, but certainly not palindromic. Less obviously, it is an example of a logarithmically concave sequence. A sequence $a_0, a_1, \ldots, a_d$ of real numbers is *logarithmically concave* provided that $a_j^2 \geq a_{j-1}a_{j+1}$ for all $1 \leq j \leq n-1$.

Exercise: Show that if (a_j) is a logarithmically concave sequence of positive real numbers, then it is a unimodal sequence.

Let $A(t) = \sum_{i=0}^{m} a_i t^i$ and $B(t) = \sum_{j=0}^{n} b_j t^j$ be polynomials with nonnegative real coefficients.

Exercise: Show that if (a_i) and (b_j) are logarithmically concave sequences then the sequence of coefficients (c_k) of $C(t) := A(t)B(t)$ is also logarithmically concave.

Exercise: Show that if (a_i) and (b_j) are palindromic unimodal sequences then (c_k) is too.

Exercise: Give an example to show that it is possible for the sequences (a_i) and (b_j) to be unimodal, but for (c_k) to fail to be unimodal.

Logarithmically concave sequences seem to crop up all over the place in certain kinds of enumeration problems. Often, the evidence is empirical and no proofs are available. One instance of this is a famous conjecture of Mason from the late 1960s. (The conjecture is made more generally for matroids – the statement below is for the special case of graphs.)

Conjecture 5.8 (Mason)**.** *Let $G = (V, E)$ be a finite connected graph (possibly with loops or multiple edges) with m edges, and for $0 \leq j \leq m$ let $q_j(G)$ be the number of connected spanning subgraphs of G with exactly j edges. Then the sequence $(q_j(G))$ is logarithmically concave.*

Verification of this conjecture for any reasonably interesting classes of graphs would be a very interesting development in enumerative graph theory.

By now you might be thinking: "Okay, fine, logarithmic concavity is interesting, but what does it have to do with representations of $\mathfrak{sl}_2(\mathbb{C})$?" I am *so glad* you asked. . . .;-)