C&O 430/630 Fall 2006 Homework 4.

1. Let $U = \bigoplus_{i\in\mathbb{Z}} U_i$ and $V = \bigoplus_{j\in\mathbb{Z}} V_j$ be complex vector spaces carrying representations of $\mathfrak{sl}(2,\mathbb{C})$ (where the U_i and V_j are the eigenspaces of H). On the tensor product space $U \otimes V$ define endomorphisms X and Y by

$$\mathsf{X}(\mathbf{u}\otimes\mathbf{v}) := (\mathsf{X}\mathbf{u})\otimes\mathbf{v} + \mathbf{u}\otimes(\mathsf{X}\mathbf{v})$$

and

$$\mathsf{Y}(\mathbf{u}\otimes\mathbf{v}) := (\mathsf{Y}\mathbf{u})\otimes\mathbf{v} + \mathbf{u}\otimes(\mathsf{Y}\mathbf{v})$$

for all pure tensors, extended linearly, and let $\mathsf{H} := \mathsf{XY} - \mathsf{YX}$.

Show that $\{\mathsf{X}, \mathsf{Y}, \mathsf{H}\}$ spans a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $U \otimes V$.

2. Let V be a finite dimensional complex vector space, written as a direct sum of subspaces $V = \bigoplus_{j\in\mathbb{Z}} V_j$. Assume that U, D, and $\Lambda := \mathsf{UD} - \mathsf{DU}$ are endomorphisms of V with the following properties:
* for all $j \in \mathbb{Z}$, $\mathsf{U} : V_j \longrightarrow V_{j+2}$;
* for all $j \in \mathbb{Z}$, $\mathsf{D} : V_j \longrightarrow V_{j-2}$;
* for all $j \in \mathbb{Z}$, there is a real number λ_j such that $\Lambda\mathbf{v} = \lambda_j\mathbf{v}$ for all $\mathbf{v} \in V_j$.

Show that if $\lambda_j > 0$ for all $j > 0$, and $\lambda_{-j} = -\lambda_j$ for all $j \in \mathbb{Z}$, then there is a representation of $\mathfrak{sl}(2,\mathbb{C})$ on V such that for all $j \in \mathbb{Z}$, V_j is the j-th eigenspace of $\mathsf{H} = \mathsf{XY} - \mathsf{YX}$.

3. **(a)** Use the Matrix–Tree Theorem to show that for each $n \geq 1$, the number of trees with vertex–set $\{1, 2, \ldots, n\}$ is n^{n-2}.

(b) Use the Matrix–Tree Theorem to determine the number of spanning trees of the complete bipartite graph $K_{a,b}$ (for all $a \geq 1$ and $b \geq 1$).