C&O 430/630 Fall 2006 Homework 3.

Due Wednesday, November 15th at the beginning of class.

C&O 430: answer four questions, including at least one of 5 or 6.
C&O 630: answer all questions.

1(a) Prove the identity $P(-t) = E'(t)/E(t)$ for the generating functions of one-part power-sum and elementary symmetric functions, without using the fact that $P(t) = H'(t)/H(t)$.

1(b) Use part (a) to prove that for any partition λ,

$$\omega(p_\lambda) = (-1)^{|\lambda|+\ell(\lambda)} p_\lambda.$$

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2. From the identity $P(-t) = E'(t)/E(t)$ for the generating functions of one-part power-sum and elementary symmetric functions, prove that for all $n \in \mathbb{N}$:

$$p_n = \det \begin{bmatrix} e_1 & 1 & 0 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & 0 & \dots & 0 \\ 3e_3 & e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ (n-1)e_{n-1} & e_{n-2} & e_{n-3} & e_{n-4} & \dots & 1 \\ ne_n & e_{n-1} & e_{n-2} & e_{n-3} & \dots & e_1 \end{bmatrix}.$$

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3. Prove the dual form of the Jacobi-Trudi Formula: for any partition λ,

$$s_\lambda = \det(e_{\lambda'_i - i + j}).$$

(over...)

4. Let λ and μ be partitions such that $F_\mu \subseteq F_\lambda$. A *skew tableau of shape* λ/μ is a function $T : F_\lambda \smallsetminus F_\mu \longrightarrow \mathbb{P}$ which is weakly increasing from left to right along rows, and strictly increasing from top to bottom along columns. The *skew Schur function of shape* λ/μ is $s_{\lambda/\mu} := \sum_T \mathbf{x}^T$, with the sum over all skew tableau of shape λ/μ.

(a) *Briefly* sketch a proof that $s_{\lambda/\mu}$ is a symmetric function.

(b) Derive a formula for $s_{\lambda/\mu}$ as a polynomial in the complete symmetric functions.

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5. For any $m, n \in \mathbb{N}$, consider the $(m+1)$-by-$(m+1)$ determinant

$$D(m,n) := \det\left(\binom{n}{i+j}\right)$$

in which $0 \leq i, j \leq m$. For each $m, n \in \mathbb{N}$, determine whether $D(m,n)$ is positive, negative, or zero. [Hint: interpret the binomial coefficients as evaluations of symmetric functions.]

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6. Prove that for any partition λ and positive integer r,

$$s_\lambda h_r = \sum_\mu s_\mu,$$

in which the sum is over all partitions μ such that $|\mu| = |\lambda| + r$ and $F_\mu \smallsetminus F_\lambda$ has at most one box per column.
