UNIVERSITY OF WATERLOO
PRACTICE QUESTIONS
WINTER TERM 2009

Surname: ____________________

First Name: ____________________

Id.#: ____________________

Course Number	CO 220
Course Title	Introduction to Combinatorics
Instructor	David Wagner
Date of Exam	April 23, 2009
Time Period	
Number of Exam Pages (including this cover sheet)	11
Exam Type	Closed Book
Additional Materials Allowed	calculators
Additional Instructions	THERE ARE MORE QUESTIONS HERE THAN WILL BE ON THE FINAL EXAM

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	xx		5	xx	
2	xx		6	xx	
3	xx		7	xx	
4	xx		8	xx	
			TOTAL	60	

1. Consider the generating function

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1 + x - 5x^2}{(1 - 2x + x^2)(1 - 2x)} = 1 + 5x + 10x^2 + 17x^3 + \cdots$$

(a) Give a recurrence relation and initial conditions that determine the values of c_n for all $n \geq 0$.

(b) Use your recurrence from part (a) to calculate c_n for all $n \leq 6$.

n	0	1	2	3	4	5	6
c_n	1	5	10	17			

(c) With the same generating function as above,

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1+x-5x^2}{(1-2x+x^2)(1-2x)} = 1 + 5x + 10x^2 + 17x^3 + \cdots$$

use partial fractions to derive a formula for c_n as a function of $n \geq 0$.

2(a) What is the probability that if you are randomly dealt 5 cards from a standard deck, then at least 3 of those cards will have the same suit?

(b) What is the probability that if you are randomly dealt 7 cards from a standard deck, then at least 4 of those cards will have the same suit?

(c)* What is the probability that if you are randomly dealt 7 cards from a standard deck, then at least 3 of those cards will have the same suit? [Be careful! You might get 3 hearts and 3 diamonds and a club. . .]

3. For a graph G and integer $k \geq 0$, let n_k be the number of vertices of G that have degree k.

(a) Show that if G is a tree with $p \geq 2$ vertices then $n_0 = 0$ and

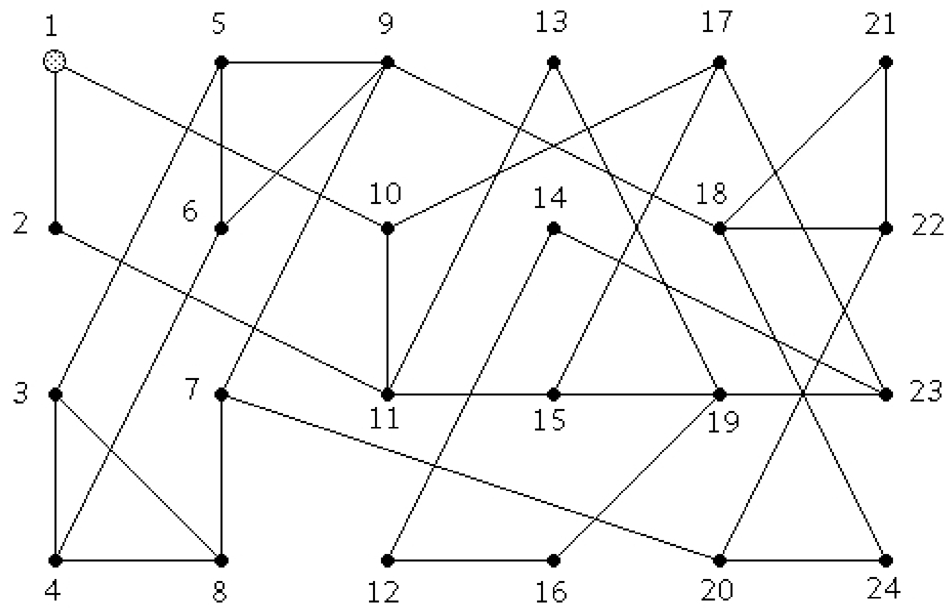
$$2 = n_1 - n_3 - 2n_4 - 3n_5 - \cdots .$$

(b) Consider a graph G with n_k given by the following table:

k	0	1	2	3	4	5
n_k	0	20	8	4	1	2

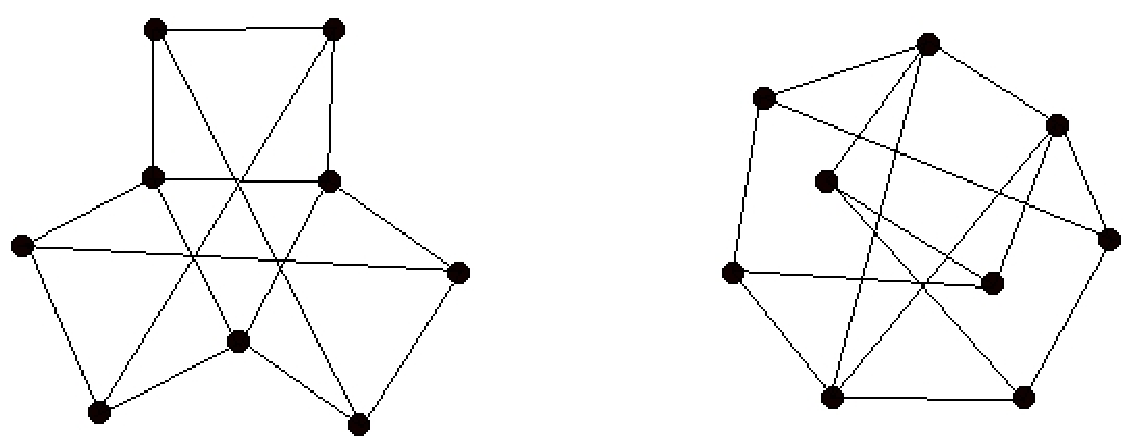
and $n_k = 0$ for all $k \geq 6$. Assume that G does not contain any cycles. How many connected components does G have?

4. Construct a Breadth-First Search Tree for the graph shown, beginning at the root vertex labelled 1. At each stage of the algorithm choose the next vertex to be the vertex labelled by the smallest number, among all the presently available vertices. Record the labels of the vertices in a queue, in the order that they join the tree, and draw the tree inside the graph.



Queue: 1

5. The two graphs shown below are isomorphic to each other. Show an isomorphism between these two graphs.



6. For each part, either draw a picture of a graph satifying the given conditions, or explain why no such graph exists.

(a) A graph with degree sequence $(2, 2, 3, 3, 3, 4, 4, 5, 5, 6, 6, 6)$.

(b) A connected planar graph with 12 vertices and 21 regions.

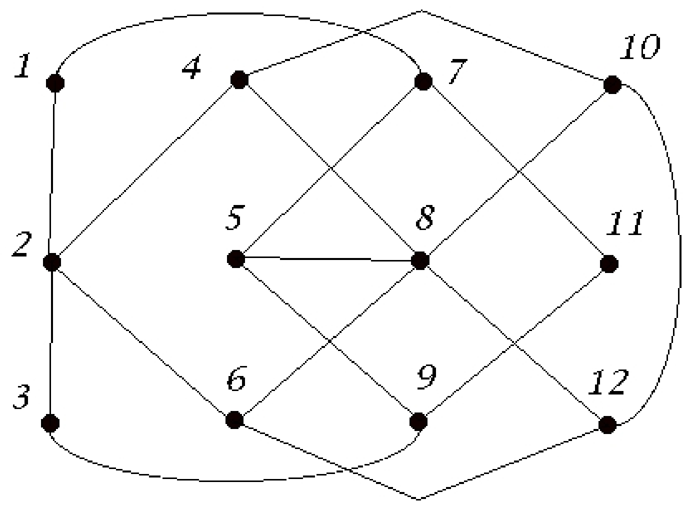
(c) A bipartite graph with 12 vertices and 37 edges.

(d) A graph in which every vertex has degree three, and which has no cycles of length 3 or 4.

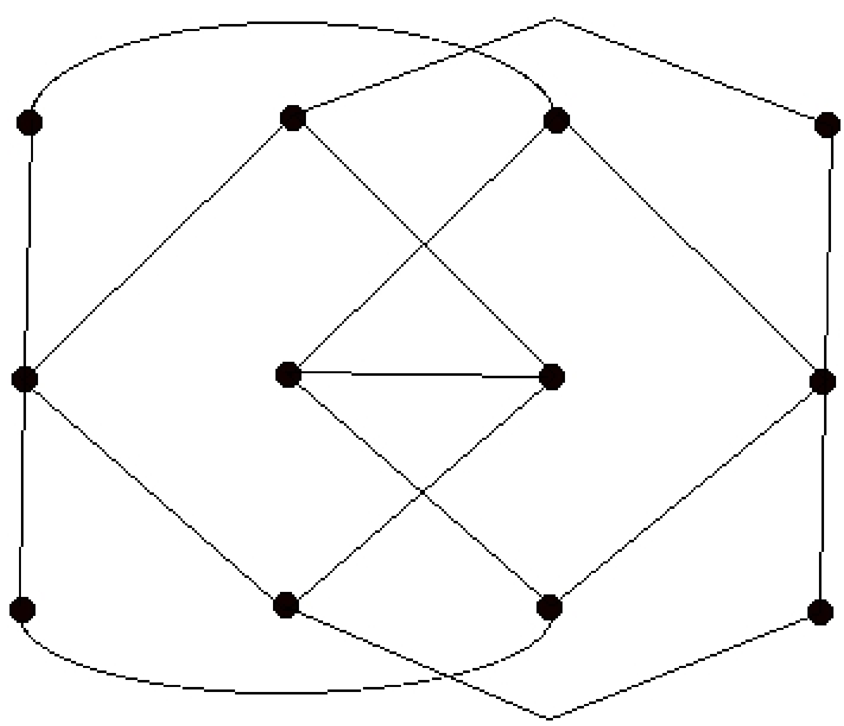
7. Prove that a graph $G = (V, E)$ is a tree if and only if for every pair of vertices $v, w \in V$, there is a **unique** path in G from v to w.

8. For each of parts (a) and (b) decide whether or not the graph shown is planar. If it is planar then draw it properly in the plane. If it is not planar then show a subdivision of K_5 or $K_{3,3}$ as a subgraph.

(a)



(b)



9. Let G be a connected planar graph with $p \geq 3$ vertices, and with n_k vertices of degree k (for each $k \geq 1$). Show that

$$5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 \geq 12 + n_7 + 2n_8 + 3n_9 \cdots .$$