CO 220 Homework #5 SOLUTIONS

1(a) *What is the maximum number of edges that a (simple) graph with p vertices can have?*

The edges are 2-element sets of vertices. There are $\binom{p}{2}$ 2-element subsets of the set of p vertices. This is the maximum number of edges in a simple graph with p vertices.

1(b) *Show that if a (simple) graph G has 17 vertices and 121 edges, then G is connected.*

If G is not connected then let H be a connected component of G, and let $J = G \setminus H$ be the rest of G. If H has h vertices then J has $17 - h$ vertices. Since H is not empty and not all of G, we have $1 \leq h \leq 16$. The maximum number of edges that G could have is now

$$\begin{aligned} q(h) &= \binom{h}{2} + \binom{17-h}{2} \\ &= \frac{h(h-1)}{2} + \frac{(17-h)(16-h)}{2} \\ &= \frac{h^2 - h + 272 - 33h + h^2}{2} \\ &= h^2 - 17h + 136. \end{aligned}$$

This function is maximized for $1 \leq h \leq 16$ either at an endpoint of the interval ($h = 1$ or $h = 16$) or at a critical point (where the derivative is zero). Set the derivative of $h^2 - 17h + 136$ to zero: $2h - 17 = 0$, so that $h = 17/2$. Now,

h	$h^2 - 17h + 136$
1	120
17/2	63.75
16	120

Thus, if G is not connected then it has at most 120 edges. Since G has 121 edges it must be connected.

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2. *Show that if $G = (V, E)$ is a graph in which every vertex has degree at least 2, then G contains a cycle C as a subgraph.*

Suppose that G does not contain a cycle. Consider any connected component H of G. Since every vertex of G has degree at least 2, H has at least three vertices. Since H is connected and contains no cycles, H is a tree. Since H is a tree with at least two vertices, H has at least two vertices of degree one. But this contradicts the hypothesis that every vertex of G has degree at least two. This contradiction shows that G must contain a cycle. (In fact, every connected component of G must contain a cycle.)

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3. *Let G be a connected graph with $2k \geq 2$ vertices of odd degree. Show that there are k trails W_1, W_2,..., W_k such that each edge of G is in exactly one of these trails.*

Let the vertices of odd degree in G be $x_1, y_1, x_2, y_2, \ldots x_k, y_k$. Form a new (multi)graph H by adding in new edges $e_1, e_2, \ldots, e_k$ with e_i joining x_i and y_i for each $1 \leq i \leq k$. Every vertex of H has even degree, and H is connected (since G is a connected spanning subgraph of H). By the theorem on Euler circuits, H has an Euler circuit Q. Now delete the k edges $e_1, e_2, \ldots, e_k$ from Q. What is left over is a set of k trails W_1, W_2,..., W_k that together cover all of the edges of G, once each. That does it!

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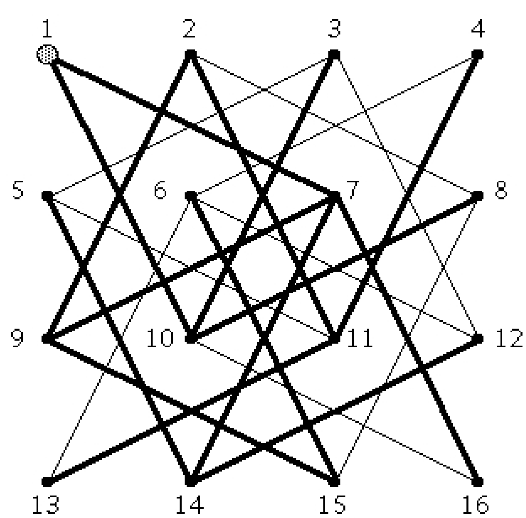
4. *The 4–by–4* Knight's Graph *is shown in the figures below.*

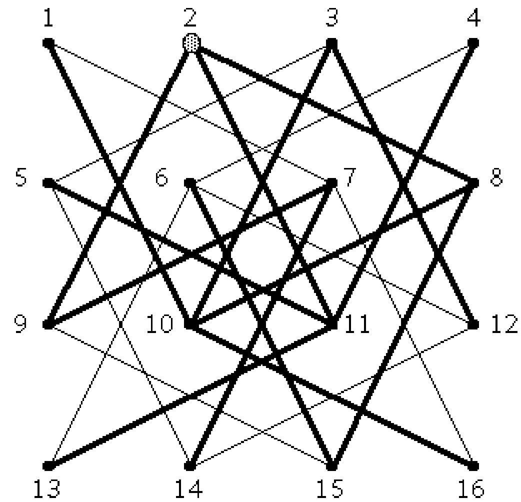
Parts (a) to (c) ask for Breadth–First Search trees in this graph with specific root vertices. When growing these trees, at each stage of the algorithm choose the next vertex to be the vertex labelled by the smallest number, among all the available vertices.

(a) *Grow a BFS tree rooted at vertex* 1.



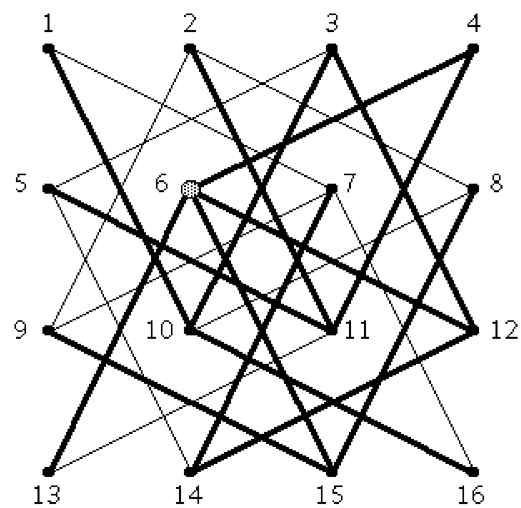
QUEUE: 1, 7, 10, 9, 14, 16, 3, 8, 2, 15, 5, 12, 11, 6, 4, 13

(b) *Grow a BFS tree rooted at vertex* 2.



QUEUE: 2, 8, 9, 11, 10, 15, 7, 4, 5, 13, 1, 3, 16, 6, 14, 12

(c) *Grow a BFS tree rooted at vertex* 6.



QUEUE: 6, 4, 12, 13, 15, 11, 3, 14, 8, 9, 2, 5, 10, 7, 1, 16

(d) *What is the maximum distance between any two vertices in this graph? Explain.*

In the BFS tree rooted at 1 the last vertex on the queue is 13. The distance from 13 to 1 is 5.

In the BFS tree rooted at 2 the last vertex on the queue is 12. The distance from 12 to 2 is 4.

In the BFS tree rooted at 6 the last vertex on the queue is 16. The

distance from 16 to 6 is 4.
Every vertex of the graph is equivalent to one of the vertices 1, 2, or 6 by a symmetry of the graph. Therefore, the maximum distance between any two vertices of this graph is 5.

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5. *For a graph G and integer $k \geq 0$, let n_k be the number of vertices of G that have degree k.*
(a) *Show that if G is a tree with $p \geq 2$ vertices then $n_0 = 0$ and*

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots .$$

Since G is connected and $p \geq 2$, G can have no vertices of degree zero: $n_0 = 0$. The total number of vertices is

$$p = n_1 + n_2 + n_3 + n_4 + n_5 + \cdots$$

By the Handshake Lemma, twice the number of edges is

$$2q = n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + \cdots$$

Since G is a tree, $q = p - 1$. That is, $2q = -2 + 2p$, so that

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + \cdots = -2 + 2n_1 + 2n_2 + 2n_3 + 2n_4 + 2n_5 + \cdots$$

Rearrange this to get

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots$$

as desired.
(b) *Consider a graph G with n_k given by the following table:*

k	0	1	2	3	4	5
n_k	0	20	8	4	1	2

and $n_k = 0$ for all $k \geq 6$. Assume that G does not contain any cycles. How many connected components does G have?

The graph G has

$$p = 20 + 8 + 4 + 1 + 2 = 35$$

vertices. By the Handshake Lemma,

$$\begin{aligned} 2q &= 20 + 2 \cdot 8 + 3 \cdot 4 + 4 \cdot 1 + 5 \cdot 2 \\ &= 20 + 16 + 12 + 4 + 10 = 62 \end{aligned}$$

so that G has $q = 31$ edges.

If G contains no cycles then every connected component of G is a tree. Let the components of G be G_1, G_2, $\ldots G_c$. If G_i has p_i vertices and q_i edges then $q_i = p_i - 1$, since G_i is a tree. That is, $p_i - q_i = 1$.

Now, $p = p_1 + p_2 + \cdots + p_c$ and $q = q_1 + q_2 + \cdots + q_c$. Therefore,

$$p - q = (p_1 - q_1) + (p_2 - q_2) + \cdots + (p_c - q_c) = c.$$

Since $p = 35$ and $q = 31$ we see that G has $c = p - q = 35 - 31 = 4$ connected components.

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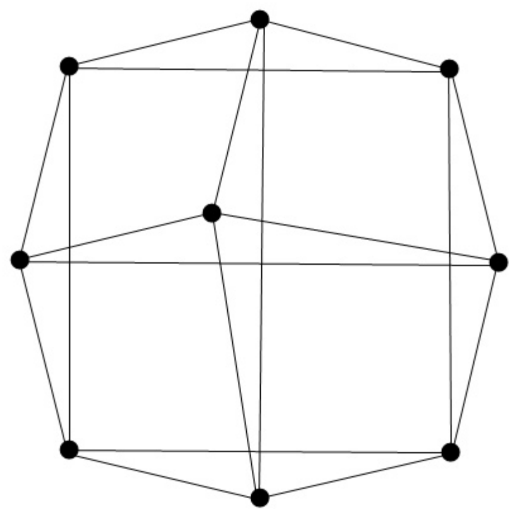
6. *Is the graph shown below planar or nonplanar? Explain your answer.*



MY BAD! I didn't have enough time to get to this material. This question will not be graded this week, but it will reappear as question one on the sixth homework assignment.
