CO 220 Homework #3 SOLUTIONS

---

1. *How many ways are there to roll a total of 25 on five six-sided dice? (Show your calculation.)*

The generating function for the number of dots on top of one die is

$$x + x^2 + x^3 + x^4 + x^5 + x^6 = \frac{x(1-x^6)}{1-x}.$$

The generating function for the number of dots on top of five dice is thus

$$\left(\frac{x(1-x^6)}{1-x}\right)^5.$$

To answer the question we want the coefficient of x^{25} in this expression:

$$\begin{aligned}
[x^{25}]\left(\frac{x(1-x^6)}{1-x}\right)^5 &= [x^{25}]x^5 \cdot (1-x^6)^5 \cdot \frac{1}{(1-x)^5} \\
&= [x^{20}] \cdot (1-x^6)^5 \cdot \frac{1}{(1-x)^5} \\
&= [x^{20}]\left(\sum_{j=0}^{5}\binom{5}{j}(-x^6)^j\right)\left(\sum_{n=0}^{\infty}\binom{n+4}{4}x^n\right) \\
&= [x^{20}](1-5x^6+10x^{12}-10x^{18}+5x^{24}-x^{30})\left(\sum_{n=0}^{\infty}\binom{n+4}{4}x^n\right) \\
&= [x^{20}](1-5x^6+10x^{12}-10x^{18})\left(\sum_{n=0}^{\infty}\binom{n+4}{4}x^n\right) \\
&= \binom{20+4}{4} - 5\binom{14+4}{4} + 10\binom{8+4}{4} - 10\binom{2+4}{4} \\
&= \binom{24}{4} - 5\binom{18}{4} + 10\binom{12}{4} - 10\binom{6}{4} \\
&= 10626 - 5 \cdot 3060 + 10 \cdot 495 - 10 \cdot 15 \\
&= 126.
\end{aligned}$$

---

2. *Let a_n be the number of compositions of size n in which each part is at least three.*

(a) *Show that*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1-x}{1-x-x^3}.$$

The allowed sizes for a single part are given by the set $\{3, 4, 5, ...\}$ of integers that are at least 3. The generating function for one part in this set is

$$x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1-x}.$$

The generating function for a partition with k parts in this set is thus

$$\left(\frac{x^3}{1-x}\right)^k.$$

Since these partitions can have any number of parts, k can be any nonnegative integer, so the generating function for all these partitions is

$$\sum_{k=0}^{\infty} \left(\frac{x^3}{1-x}\right)^k = \frac{1}{1-x^3/(1-x)} = \frac{1-x}{1-x-x^3},$$

as claimed.

(b) *Show that $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, and for all $n \geq 3$, $a_n = a_{n-1} + a_{n-3}$.*

From the form of the generating function we see that for all $n \in \mathbb{N}$,

$$a_n - a_{n-1} + a_{n-3} = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2, \end{cases}$$

with the convention that $a_n = 0$ if $n < 0$. Thus we see that $a_0 = 1$, $a_1 - a_0 = -1$, $a_2 - a_1 = 0$, and $a_n = a_{n-1} + a_{n-3}$ if $n \geq 3$. This yields the initial conditions and recurrence relation claimed.

(c) *Compute a_n for all $0 \leq n \leq 16$.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	1	0	0	1	1	1	2	3	4	6	9	13	19	28	41	60	88

---

3. *Let b_n be the number of compositions of size n in which each part is at least two, and which have an odd number of parts.*

(a) *Show that*

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = \frac{x^2 - x^3}{1 - 2x + x^2 - x^4}.$$

The allowed sizes for a single part are given by the set $\{2, 3, 4, ...\}$ of integers that are at least 2. The generating function for one part in this set is

$$x^2 + x^3 + x^4 + \cdots = \frac{x^2}{1-x}.$$

The generating function for a partition with k parts in this set is thus

$$\left(\frac{x^2}{1-x}\right)^k.$$

Since these partitions can have any **odd** number of parts, k can be any odd nonnegative integer – that is, any integer of the form $k = 2j + 1$ for some nonnegative integer j. Thus, the generating function for all these partitions is

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\frac{x^2}{1-x}\right)^{2j+1} &= \frac{\left(\frac{x^2}{1-x}\right)}{1 - \left(\frac{x^2}{1-x}\right)^2} \\
&= \frac{x^2(1-x)}{(1-x)^2 - x^4} \\
&= \frac{x^2 - x^3}{1 - 2x + x^2 - x^4},
\end{aligned}$$

as claimed.

(b) *Derive initial conditions and a recurrence relation that determines the sequence (b_n) for all $n \geq 0$.*

From the form of the generating function we see that for all $n \in \mathbb{N}$,

$$b_n - 2b_{n-1} + b_{n-2} - b_{n-4} = \begin{cases} 0 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ -1 & \text{if } n = 3, \\ 0 & \text{if } n \geq 4, \end{cases}$$

with the convention that $b_n = 0$ if $n < 0$. Thus we see that $b_0 = 0$, $b_1 - 2b_0 = 0$, $b_2 - 2b_1 + b_0 = 1$, $b_3 - 2b_2 + b_1 = -1$, and $b_n = 2b_{n-1} - b_{n-2} + b_{n-4}$ if $n \geq 4$. This yields the initial conditions $b_0 = 0$, $b_1 = 0$, $b_2 = 1$, and $b_3 = 1$, and the recurrence relation $b_n = 2b_{n-1} - b_{n-2} + b_{n-4}$ if $n \geq 4$.

(c) *Compute b_n for all $0 \leq n \leq 12$.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12
b_n	0	0	1	1	1	1	2	4	7	11	17	27	44

---

4. *Consider the power series*

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{1 - 4x + 5x^2}{1 - 4x + 5x^2 - 2x^3} = 1 + 2x^3 + 8x^4 + 22x^5 + 52x^6 + \cdots .$$

(a) *Derive initial conditions and a recurrence relation that determines the sequence (c_n) for all $n \geq 0$. (You don't need to compute the sequence.)*

From the form of the power series we see that for all $n \in \mathbb{N}$,

$$c_n - 4c_{n-1} + 5c_{n-2} - 2c_{n-3} = \begin{cases} 1 & \text{if } n = 0, \\ -4 & \text{if } n = 1, \\ 5 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$

with the convention that $c_n = 0$ if $n < 0$. Thus we see that $c_0 = 1$, $c_1 - 4c_0 = -4$, $c_2 - 4c_1 + 5c_0 = 5$, and $c_n = 4c_{n-1} - 5c_{n-2} + 2c_{n-3}$ if $n \geq 3$. This yields the initial conditions $c_0 = 1$, $c_1 = 0$, and $c_2 = 0$, and the recurrence relation $c_n = 4c_{n-1} - 5c_{n-2} + 2c_{n-3}$ if $n \geq 3$.

For completeness (although this is not required) here are the first few values in the sequence of coefficients:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
c_n	1	0	0	2	8	22	52	114	290	494	1004	2026	4072

(b) *Use Partial Fractions to obtain a formula for the coefficient c_n as a function of n. (Hint: $1 - 4x + 5x^2 - 2x^3 = (1-2x)(1-2x+x^2)$.)*

Since the degree of the numerator is strictly less than the degree of the denominator, $C(x)$ has a Partial Fractions expansion. The denominator factors as $(1-2x)(1-x)^2$. Thus, by Partial Fractions there are constants K, L, and M such that

$$C(x) = \frac{1-4x+5x^2}{1-4x+5x^2-2x^3} = \frac{K}{1-2x} + \frac{L}{1-x} + \frac{M}{(1-x)^2}.$$

Clear the denominator to obtain

$$1 - 4x + 5x^2 = K(1-x)^2 + L(1-2x)(1-x) + M(1-2x).$$

At $x = 1$ this equation gives $2 = -M$, so that $M = -2$.
At $x = 1/2$ this equation gives $1/4 = K/4$, so that $K = 1$.
At $x = 0$ this equation gives $1 = K + L + M$, so that

$$L = 1 - K - M = 1 - 1 + 2 = 2.$$

Therefore,

$$\begin{aligned} C(x) &= \frac{1-4x+5x^2}{1-4x+5x^2-2x^3} \\ &= \frac{1}{1-2x} + \frac{2}{1-x} - \frac{2}{(1-x)^2} \\ &= \sum_{n=0}^{\infty} 2^n x^n + 2\sum_{n=0}^{\infty} x^n - 2\sum_{n=0}^{\infty} \binom{n+1}{1} x^n \\ &= \sum_{n=0}^{\infty} (2^n - 2n)\, x^n. \end{aligned}$$

The coefficient of x^n in $C(x)$ is thus $c_n = 2^n - 2n$ for all $n \in \mathbb{N}$.

5. *Consider the power series*

$$G(x) = \sum_{n=0}^{\infty} g_n x^n = \frac{x + 7x^2}{1 - 3x^2 - 2x^3} = x + 7x^2 + 3x^3 + 23x^4 + 23x^5 + 75x^6 + \cdots .$$

Use Partial Fractions to obtain a formula for the coefficient g_n as a function of n.

Since the degree of the numerator is strictly less than the degree of the denominator, $G(x)$ has a Partial Fractions expansion. The denominator takes the value 0 when $x = -1$: this means that $1+x$ is a factor of the denominator. Doing the long division (or otherwise) we see that

$$1 - 3x^2 - 2x^3 = (1 - x - 2x^2)(1 + x).$$

Again, $1 - x - 2x^2$ vanishes when $x = -1$, so that $1 + x$ is a factor of this: in fact

$$1 - x - 2x^2 = (1 - 2x)(1 + x).$$

Finally, the denominator is completely factored:

$$1 - 3x^2 - 2x^3 = (1 - 2x)(1 + x)^2.$$

Thus, by Partial Fractions there are constants K, L, and M such that

$$G(x) = \frac{x + 7x^2}{1 - 3x^2 - 2x^3} = \frac{K}{1 - 2x} + \frac{L}{1 + x} + \frac{M}{(1 + x)^2}.$$

Clear the denominator to obtain

$$x + 7x^2 = K(1 + x)^2 + L(1 - 2x)(1 + x) + M(1 - 2x).$$

At $x = -1$ this equation gives $6 = 3M$, so that $M = 2$.
At $x = 1/2$ this equation gives $9/4 = 9K/4$, so that $K = 1$.
At $x = 0$ this equation gives $0 = K + L + M$, so that

$$L = -K - M = -1 - 2 = -3.$$

Therefore,

$$
\begin{aligned}
G(x) &= \frac{x + 7x^2}{1 - 3x^2 - 2x^3} \\
&= \frac{1}{1-2x} + \frac{-3}{1+x} + \frac{2}{(1+x)^2} \\
&= \sum_{n=0}^{\infty} 2^n x^n - 3\sum_{n=0}^{\infty}(-1)^n x^n + 2\sum_{n=0}^{\infty}\binom{n+1}{1}(-1)^n x^n \\
&= \sum_{n=0}^{\infty}\left(2^n - 3(-1)^n + 2(n+1)(-1)^n\right)x^n \\
&= \sum_{n=0}^{\infty}\left(2^n + (2n-1)(-1)^n\right)x^n.
\end{aligned}
$$

The coefficient of x^n in $G(x)$ is thus $g_n = 2^n + (2n-1)(-1)^n$ for all $n \in \mathbb{N}$.