CO 220 Practice Questions
Inclusion/Exclusion
Solutions

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1. *Consider the set of all permutations of* $\{a, b, c, d, e, f, g\}$ *that do not contain any of the sequences* ab*, or* bcd*, or* be *as substrings. Show that there are* 3620 *such permutations.*

There are $7! = 5040$ permutations of the 7-element set $U = \{a, b, c, d, e, f, g\}$. To count those permutations that **do** contain at least one of these sequences, for each nonempty subset S of $\{ab, bcd, be\}$ let A_S be the set of permutations of U that contain all the sequences in S. The sizes of these sets are given in the following table (and the signs come from the Inclusion/Exclusion formula):

S	$\pm\lvert A_S\rvert$
$\{ab\}$	$+6!$
$\{bcd\}$	$+5!$
$\{be\}$	$+6!$
$\{ab, bcd\}$	$-4!$
$\{ab, be\}$	$-5!$
$\{bcd, be\}$	-0
$\{ab, bcd, be\}$	$+0$

As examples, the permuations that contain ab are basically permutations of the 6-element set $\{ab, c, d, e, f, g\}$; the permuations that contain bcd are basically permutations of the 5-element set $\{a, bcd, e, f, g\}$; and the permuations that contain ab and bcd are basically permutations of the 4-element set $\{abcd, e, f, g\}$.

Thus, by Inclusion/Exclusion, the number of permutations of U that **do** contain one of the sequences ab, or bcd, or be as a substring is

$$\begin{aligned} & 6! + 5! + 6! - 4! - 5! \\ = \; & 720 + 120 + 720 - 24 - 120 \\ = \; & 1440 - 24 = 1416. \end{aligned}$$

Therefore, the number of permutations of $\{a, b, c, d, e, f, g\}$ that **do not** contain any of these substrings is

$$7! - 1416 = 5040 - 1416 = 3624.$$

(OOPS! Sorry, it looks like there was a typo in the question... I hope you figured that out.)

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2. *Consider the set of all* 10*-element multisets with elements of* 5 *types, in which each type of element occurs at most* 3 *times. Show that there are* 101 *such multisets.*
(Hint: For each $i \in \{1, 2, 3, 4, 5\}$ *let* A_i *be the set of* 10*-element* 5*-type multisets in which the* i*-th type occurs* ***at least*** 4 *times. What is* $|A_1 \cup \cdots \cup A_5|$*?.)*

A 10-element multiset with elements of 5 types is a sequence $(m_1, m_2, m_3, m_4, m_5)$ of natural numbers such that $m_1 + m_2 + m_3 + m_4 + m_5 = 10$. There are $\binom{10+5-1}{5-1} = \binom{14}{4} = 1001$ of these.

If m_1 occurs at least 4 times then $(m_1 - 4, m_2, m_3, m_4, m_5)$ is a 6-element multiset with elements of 5 types. There are $\binom{6+5-1}{5-1} = \binom{10}{4} = 210$ of these, so that $|A_1| = 210$. Similarly, $|A_i| = 210$ for each $i \in \{1, 2, 3, 4, 5\}$.

If both m_1 and m_2 occur at least 4 times then $(m_1 - 4, m_2 - 4, m_3, m_4, m_5)$ is a 2-element multiset with elements of 5 types. There are $\binom{2+5-1}{5-1} = \binom{6}{4} = 15$ of these, so that $|A_{\{1,2\}}| = 15$. Similarly, $|A_{\{i,j\}}| = 15$ for every 2-element subset of $\{1, 2, 3, 4, 5\}$.

No more than two of the m_i can be at least 4 at the same time, since the multiset only has 10 elements. Therefore $|A_S| = 0$ if S has three or more elements.

By Inclusion/Exclusion, the number of 10-element 5-type multisets in which

some type occurs at least 4 times is:

$$\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| &= \sum_{\varnothing \neq S \subseteq \{1,2,3,4,5\}} (-1)^{|S|-1} |A_S| \\
&= \binom{5}{1} \cdot 210 - \binom{5}{2} \cdot 15 \\
&= 5 \cdot 210 - 10 \cdot 15 = 1050 - 150 = 900.
\end{aligned}$$

Therefore, the number of 10-element 5-type multisets in which every type occurs at most 3 times is

$$1001 - 900 = 101,$$

as claimed.