CO 220 Homework Assignment #2 Solutions
Friday, January 30th, 2009.

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1. *In this question one is dealt a hand of 5 cards from a standard deck of 52 cards (4 suits, Ace to King), which we call a* poker hand. *Briefly explain your answer to each part.*

(a) *How many poker hands are there in total?*

Since there are 52 cards in the deck and 5 cards are dealt in a poker hand, there are $\binom{52}{5} = 2598960$ possible poker hands.

(b) *How many poker hands have one pair (but no better)?*

There are 13 ranks of card (from Ace to King), from which we choose one for the rank of the pair. There are 4 cards of that rank, from which we choose 2 to be the paired cards. There are 12 other ranks, from which we choose 3 to be the ranks of the other three cards. (This way we cannot get two pair, or a straight or flush...). For each of these three cards there are 4 choices for the suit. The total number of choices is thus

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1098240.$$

(c) *How many poker hands have two pairs (but no better)?*

There are 13 ranks of card (from Ace to King), from which we choose 2 ranks for the pairs in the hand. For both of these pairs, we choose two of the four cards of that rank. The fifth card can't have either of the ranks of the pairs (or we'd have a full house) so there are $52 - 8 = 44$ choices for the fifth card. In total the number of choices is

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 = 78 \cdot 6 \cdot 6 \cdot 44 = 123552.$$

(d) *How many poker hands have three of a kind (but no better)?*

There are 13 choices for the rank of the triple, and $\binom{4}{3}$ choices for which three cards of this rank are held. Then there are $\binom{12}{2}$ choices for the ranks of the other two cards (they can't have the same rank, or we'd have a full house) and 4 choices for the suit of each of these. In total the number of choices is

$$13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4 \cdot 4 = 13 \cdot 4 \cdot 66 \cdot 4 \cdot 4 = 54912.$$

(e) *How many poker hands are straight flushes?*

There are 10 choices for the smallest rank in the straight, and 4 choices for the suit of the flush. This determines all the cards in the straight flush, so the total number of straight flushes is 40.

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2. *In* Draw Poker *one is dealt a poker hand and then has an opportunity to exchange some (or none, or all) of the cards. Suppose you are dealt the hand* $8\spadesuit\ 8\diamondsuit\ 8\clubsuit\ A\heartsuit\ 5\diamondsuit$.

(a) *If you throw away the* $5\diamondsuit$ *and draw one card, what is the probability that your hand improves (to Full House or Quads)?*

There are 47 cards not in your hand – each of them is equally likely to be drawn. The cards that help your hand are $8\heartsuit$, $A\diamondsuit$, $A\clubsuit$, and $A\spadesuit$. Thus, the probability that your hand improves is $4/47 = 0.085106$ (to six decimal places), about 8.5%.

(b) *If you throw away both* $A\heartsuit\ 5\diamondsuit$ *and draw two cards, what is the probability that your hand improves (to Full House or Quads)?*

There are 47 cards not in your hand – each of the $\binom{47}{2} = 1081$ 2-element subsets of them is equally likely to be drawn. There are 46 of these draws that contain the $8\heartsuit$. There are $10\binom{4}{2} + 2\binom{3}{2} = 66$ of these draws that are pairs: 10 ranks have 4 cards left (every rank except A, 5, or 8) and 2 ranks have 3 cards left (namely A or 5). Of the remaining cards of that rank, you choose two of them when you draw a pair. The total number of 2-card draws that help your hand is thus $46 + 66 = 112$. Therefore, in this case the probability that your hand improves is $112/1081 = 0.103608$ (to six decimal places), about 10.4%.

(Drawing two cards is clearly better.)

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3. *Prove that for any integers $n \geq 1$ and $t \geq 2$:*

$$\binom{n+t-1}{t-1} = \sum_{k=0}^{n} \binom{n-k+t-2}{t-2}.$$

The LHS is the number of multisets of size n with elements of t types. Remember that such a multiset is a t-tuple $(m_1, m_2, \ldots, m_t)$ of natural numbers such that $m_1 + m_2 + \cdots + m_t = n$. The value of m_t can be any integer k in the range $0 \leq k \leq n$. If $m_t = k$ then $(m_1, m_2, \ldots, m_{t-1})$ adds up to $m_1 + m_2 + \cdots + m_{t-1} = n - k$. That is, these $t-1$ numbers form a multiset of size $n-k$ with elements of $t-1$ types. There are $\binom{n-k+t-2}{t-2}$ such multisets. This is also the number of multisets of size n with elements of t types in which $m_t = k$. Since k can be any integer in the range $0 \leq k \leq n$, the total number of multisets of size n with elements of t types is

$$\binom{n+t-1}{t-1} = \sum_{k=0}^{n} \binom{n-k+t-2}{t-2}.$$

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4. *For each $t \geq 1$, let $f(t)$ denote the probability that a randomly chosen $(2t)$–element multiset with elements of t types is such that every type of element occurs at least once.*

(a) *Give a formula for $f(t)$ as a function of t.*

The total number of of multisets of size $n = 2t$ with elements of t types is

$$\binom{n+t-1}{t-1} = \binom{3t-1}{t-1} = \frac{(3t-1)!}{(t-1)!(2t)!}.$$

Remember that such a multiset is a t-tuple $(m_1, m_2, \ldots, m_t)$ of natural numbers such that $m_1 + m_2 + \cdots + m_t = 2t$. If every type of element occurs at least once then $m_i \geq 1$ for all $1 \leq i \leq t$. In this case, put $k_i = m_i - 1$ for each $1 \leq i \leq t$. Then $(k_1, k_2, \ldots, k_t)$ is a multiset of size t with elements of t types. There are

$$\binom{t+t-1}{t-1} = \binom{2t-1}{t-1} = \frac{(2t-1)!}{(t-1)!(t)!}$$

such multisets. These correspond to (2t)-element multisets with t types, in which each type occurs at least once.

Therefore, the probability $f(t)$ is

$$f(t) = \frac{\binom{2t-1}{t-1}}{\binom{3t-1}{t-1}} = \frac{(2t-1)!(2t)!}{(3t-1)!t!} = \frac{(2t)(2t-1)\cdots(t+1)}{(3t-1)(3t-2)\cdots(2t)}.$$

(b) *Compute $f(t)$ to 6 decimal places for each $1 \le t \le 6$.*

t	$f(t)$	$f(t)$
1	$2/2$	1.000000
2	$4 \cdot 3/5 \cdot 4$	0.600000
3	$6 \cdot 5 \cdot 4/8 \cdot 7 \cdot 6$	0.357143
4	$8 \cdot 7 \cdot 6 \cdot 5/11 \cdot 10 \cdot 9 \cdot 8$	0.212121
5	$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6/14 \cdot 13 \cdot 12 \cdot 11 \cdot 10$	0.125874
6	$12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7/17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$	0.074661

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5. *Consider a randomly chosen permutation σ of the set $\{a, b, c, d, e, f, g, h\}$.*

(The total number of permutations of this set is $8! = 40320$.)

(a) *What is the probability that in σ the letters a and b are* ***not*** *adjacent? In other words, what is the probability that σ does not contain either ab or ba as a substring?*

Notice that σ can **not** contain both substrings ab and ba simultaneously. There are 7! permutations containing ab (corresponding to the permutations of $\{ab, c, d, e, f, g, h\}$) and there are 7! permutations containing ba (corresponding to the permutations of $\{ba, c, d, e, f, g, h\}$). The number of permutations which do not contain either of these substrings is thus

$$8! - 7! - 7! = 40320 - 5040 - 5040 = 30240.$$

The probability in question is thus $30240/40320 = 3/4 = 0.75$.

(b) *What the probability that in σ the letters a and b are not adjacent, and the letters b and c are not adjacent?*

The permutation σ fails this condition if and only if σ contains any one (or more) of the substrings ab, ba, bc, or cb. It might contain both ab and bc (exactly when it contains abc), and it might contain both ba and cb (exactly when it contains cba). We can count permutations containing at least one of the substrings ab, ba, bc, or cb by Inclusion/Exclusion:

S	$\pm\|A_S\|$
$\{ab\}$	$+7!$
$\{ba\}$	$+7!$
$\{bc\}$	$+7!$
$\{cb\}$	$+7!$
$\{ab, ba\}$	-0
$\{ab, bc\}$	$-6!$
$\{ab, cb\}$	-0
$\{ba, bc\}$	-0
$\{ba, cb\}$	$-6!$
$\{bc, cb\}$	-0
$\{ab, ba, bc\}$	$+0$
$\{ab, ba, cb\}$	$+0$
$\{ab, bc, cb\}$	$+0$
$\{ba, bc, cb\}$	$+0$
$\{ab, ba, bc, cb\}$	-0

Thus, the number of permutations containing at least one of the substrings ab, ba, bc, or cb is

$$\begin{aligned} 4 \cdot 7! - 2 \cdot 6! &= 4 \cdot 5040 - 2 \cdot 720 \\ &= 20160 - 1440 = 18720. \end{aligned}$$

Therefore, the number of permutations that do not conatin any of the substrings ab, ba, bc, or cb is

$$8! - 18720 = 40320 - 18720 = 21600.$$

The probability in question is thus $21600/40320 = 0.535714$.