CO 220 Winter 2009 HW Solutions #1

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1. *There are* 6^4 *possible outcomes when rolling four ordinary* 6*–sided dice. Count how many of these outcomes are of each of the following types:*

(a) *no pairs;*

There are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ partial permutations of $\{1, 2, ..., 6\}$ of length 4.

(b) *exactly one pair;*

There are $\binom{4}{2}$ choices for which two dice have the same value, 6 choices for this common value, 5 choices for the value on the third die, and 4 choices for the value on the fourth die. That is $6 \cdot 6 \cdot 5 \cdot 4 = 720$ in total.

(c) *exactly two pairs;*

Think of the dice as listed from left to right: □ □ □ □. There are exactly 3 ways to split these four dice into two pairs: either AABB or ABAB or ABBA. (The pattern BBAA is the same as AABB.) There are 6 choices for the common value on the A dice, and then 5 choices for the (different) common value on the B dice. That is $3 \cdot 6 \cdot 5 = 90$ in total.

(d) *three of a kind (but not four);*

There are $\binom{4}{3}$ choices for which three dice have the same value, 6 choices for this common value, and 5 choices for the different value shown on the fourth die. That is $4 \cdot 6 \cdot 5 = 120$ in total.

(e) *four of a kind.*

There are 6 choices for the common value of all 4 dice.

(As a check, notice that every outcome is of exactly one of these types, so the sum of the answers must add up to 6^4*.)*

$$360 + 720 + 90 + 120 + 6 = 1296 = 6^4, \text{ it is true.}$$

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2. *Continuing Exercise 1, say that when rolling four dice an outcome is a "run" if it shows four different but consecutive numbers. That is, the dice show* $\{1, 2, 3, 4\}$ *or* $\{2, 3, 4, 5\}$ *or* $\{3, 4, 5, 6\}$*. Is rolling a run* ***more*** *or* ***less*** *likely than rolling three (but not four) of a kind? Explain.*

There are 3 choices for the set of values showing on the dice (given in the statement of the question). For each of these sets, there are 4! ways this set of values can appear on the four dice. That is, there are $3 \cdot 4! = 3 \cdot 24 = 72$ ways to roll a run in total. Since there are 120 ways to roll three (but not four) of a kind, rolling a run is **less likely**. (Rolling a run is even less likely than rolling two pairs.)

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3. *Assume that people's birthdays are independently and uniformly distributed over the 7 days of the week from Sunday to Saturday. (This is a reasonable assumption, with the exception of twins.) Let $p(n)$ denote the probability that in a randomly chosen group of n people, at least two of them are born on the same day of the week.*
(a) *Give a formula for $p(n)$ as a function of n. (Check that your answer gives $p(2) = 1/7$.)*
Analyzing $p(n)$ directly is quite difficult, so let's think about $1 - p(n)$ instead. This is the probability that in a randomly chosen group of n people, each of them is born on a different day of the week. This model is equivalent to rolling "no pair" on a set of n seven–sided dice. There are 7^n possible outcomes for the arrangement of birthdays for n people. Of these, the number which have no two days the same (no pairs) is $7 \cdot 6 \cdot 5 \cdots (7 - n + 1)$. In summary,

$$1 - p(n) = \frac{7 \cdot 6 \cdot 5 \cdots (7 - n + 1)}{7^n}$$

so that

$$p(n) = 1 - \frac{7 \cdot 6 \cdot 5 \cdots (7 - n + 1)}{7^n}.$$

(As a check, notice that $p(2) = 1 - (7 \cdot 6)/(7 \cdot 7) = 1 - 6/7 = 1/7$.)

(b) *Complete the following table, expressing your answers both as exact fractions and as approximations to six decimal places.*

n	$p(n)$	$p(n)$
1	0/1	0.000000
2	1/7	0.142857
3	19/49	0.387755
4	223/343	0.650146
5	2041/2401	0.850062
6	16087/16807	0.957161
7	116929/117649	0.993880
$n \geq 8$	1/1	1.000000

4(a) *Show that for $n, t, k \geq 1$, the number of n–element multisets with elements of t types in which* ***exactly*** *k different types of element occur (at least once each) is $\binom{t}{k}\binom{n-1}{k-1}$.*
There are t types of elements which are available, of which we choose exactly k to occur at least once each. Choosing k of the t types explains the factor of $\binom{t}{k}$. Now we need to ensure that the k chosen types of elements occur at least once each – so our multiset must contain at

least one element of each of these k types. This uses up k of the n elements of the multiset. There are $n-k$ elements left over, which can be distributed among the k chosen types as we please. That is, it remains to choose a multiset of size $n-k$ with elements of k types. This can be done in

$$\binom{(n-k)+(k-1)}{k-1} = \binom{n-1}{k-1}$$

ways. The total number of choices is thus $\binom{t}{k}\binom{n-1}{k-1}$, as claimed.

4(b) *Show that for $n, t \geq 1$,*

$$\binom{n+t-1}{t-1} = \sum_{k=1}^{t} \binom{t}{k}\binom{n-1}{k-1}.$$

The LHS of the formula is the total number of multisets of size n with elements of t types. Each such multiset has exactly k types of elements actually occurring, for exactly one value of k in the range $1 \leq k \leq t$. For each k in this range, part (a) shows that the number of such multisets is $\binom{t}{k}\binom{n-1}{k-1}$. Since each multiset of size n with elements of t types is counted exactly once on the RHS of the formula as well as on the LHS of the formula, the two sides of the formula are equal. This proves the result.
