UNIVERSITY OF WATERLOO
CO 220 MIDTERM EXAM SOLUTIONS
WINTER TERM 2009

Surname: ______________________

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Course Number	CO 220
Course Title	Introduction to Combinatorics
Instructor	David Wagner
Date of Exam	February 11th, 2008
Time Period	4:30 - 6:30 pm
Number of Exam Pages (including this cover sheet)	7
Exam Type	Closed Book
Additional Materials Allowed	Non-programmable calculators
Additional Instructions	READ ALL THE QUESTIONS BEFORE BEGINNING! Answer in the space provided

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	7		4	6	
2	8		5	7	
3	6		6	6	
			Total	40	

1(a) [4 pts.] When rolling 5 ordinary 6-sided dice, what is the probability of rolling one pair (but no better) – 5 2 6 5 3, for example?

6 choices for the value of the paired dice;
$\binom{5}{2}$ choices for the positions of the paired dice;
5 choices for the value of the first remaining die;
4 choices for the value of the second remaining die;
3 choices for the value of the third remaining die:

$$6 \cdot \binom{5}{2} \cdot 5 \cdot 4 \cdot 3 = 6 \cdot 10 \cdot 60 = 3600$$

ways in total. Thus, the probability is

$$\frac{3600}{6^5} = \frac{100}{216} = 0.462963$$

(to six decimal places).

1(b) [3 pts.] What is the probability that a randomly chosen multiset of size $n = 8$ with elements of the $t = 5$ types (A, B, C, D, E) has **exactly** 3 elements of type A?

The total number of multisets of size 8 with elements of 5 types is

$$\binom{8+5-1}{5-1} = \binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 11 \cdot 5 \cdot 9 = 495.$$

If there are exactly 3 elements of type A then the $8 - 3 = 5$ other elements are of the remaining 4 types: they form a multiset of size 5 with elements of 4 types. There are

$$\binom{5+4-1}{4-1} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$$

such multisets. Thus, the probability in question is

$$\frac{56}{495} = 0.113131$$

(to six decimal places).

2. In the game of Bridge, the 52 cards of a standard deck are dealt out to four players so that each player gets a set of exactly 13 cards.

(a) [3 pts.] What is the total number of possible ways the cards can be dealt in Bridge? (The order of the cards in each player's hand does not matter.)

Bridge players are traditionally called North, East, South, and West (N, E, S, W).
$\binom{52}{13}$ choices for North's hand, then
$\binom{39}{13}$ choices for East's hand, then
$\binom{26}{13}$ choices for South's hand, then
$\binom{13}{13} = 1$ choice for West's hand:

$$\begin{aligned}\binom{52}{13}\binom{39}{13}\binom{26}{13} &= \frac{52!}{13!39!} \cdot \frac{39!}{13!26!} \cdot \frac{26!}{13!13!} \\ &= \frac{52!}{13!^4} \\ &= 53,644,737,765,488,792,839,237,440,000\end{aligned}$$

ways in total. (This is a HUGE number!)

(b) [5 pts.] What is the probability in Bridge that one of the players gets all four Aces?

4 choices for which player gets all four Aces;
$\binom{48}{9}$ choices for the remaining 9 cards in that player's hand;
$\binom{39}{13}$ choices for the second player's hand, then
$\binom{26}{13}$ choices for the third player's hand, then
$\binom{13}{13} = 1$ choice for the fourth player's hand:
the probability is thus

$$\begin{aligned}\frac{4 \cdot \binom{48}{9}}{\binom{52}{13}} &= \frac{4 \cdot 48!13!39!}{9!39!52!} \\ &= \frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49} \\ &= \frac{44}{4165} = .010564\end{aligned}$$

(to six decimal places).

3. For each part, give an instance of the binomial series

$$\frac{1}{(1-z)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} z^n$$

that begins as shown. For example,

$$1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + \cdots = \sum_{n=0}^{\infty} \binom{n+2}{2} (-x)^n = \frac{1}{(1+x)^3}.$$

(a) [3 pts.]

$$2 + 6x + 12x^2 + 20x^3 + 30x^4 + 42x^5 + \cdots$$

$$= 2\sum_{n=0}^{\infty} \binom{n+2}{2} x^n = \frac{2}{(1-x)^3}.$$

(b) [3 pts.]

$$3 + 6x^2 + 9x^4 + 12x^6 + 15x^8 + \cdots$$

$$= 3\sum_{n=0}^{\infty} \binom{n+1}{1} (x^2)^n = \frac{3}{(1-x^2)^2}.$$

4. [6 pts.] What is the probability that a randomly chosen permutation of $\{a, b, c, d, e, f\}$ does **not** contain any of the subsequences *abc*, *bcd*, or *dea*?

There are $6! = 720$ permutations of the 6-element set $\{a, b, c, d, e, f\}$. How many do contain at least one of these substrings? We find out by Inclusion/Exclusion:

S	$\pm\|A_S\|$	
$\{abc\}$	$+4!$	$\{abc, d, e, f\}$
$\{bcd\}$	$+4!$	$\{a, bcd, e, f\}$
$\{dea\}$	$+4!$	$\{b, c, dea, f\}$
$\{abc, bcd\}$	$-3!$	$\{abcd, e, f\}$
$\{abc, dea\}$	$-2!$	$\{deabc, f\}$
$\{bcd, dea\}$	$-2!$	$\{bcdea, f\}$
$\{abc, bcd, dea\}$	$+0$	$\varnothing$

There are

$$3 \cdot 4! - 3! - 2 \cdot 2! = 3 \cdot 24 - 6 - 4 = 72 - 10 = 62$$

permutations of $\{a, b, c, d, e, f\}$ that do contain at least one of these substrings. Thus, there are $720 - 62 = 658$ permutations that do not contain any of them. The probability that a random permutation doesn't contain any of the given substrings is thus

$$\frac{658}{720} = \frac{329}{360} = 0.913889$$

(to six decimal places).

5(a) [1 pt.] For each natural number n, how many multisets are there with n elements of the 6 types (A, B, C, D, E, F)?

$$\binom{n+6-1}{6-1} = \binom{n+5}{5}$$

5(b) [1 pt.] For each natural number k, how many multisets are there with k elements of the 3 types (A, B, C)?

$$\binom{k+3-1}{3-1} = \binom{k+2}{2}$$

5(c) [5 pts.] Explain why the following identity holds: for each natural number n,

$$\binom{n+5}{5} = \sum_{k=0}^{n} \binom{k+2}{2}\binom{n-k+2}{2}.$$

The left-hand side is explained in part (a): it is the number of n-element multisets with 6 types of elements (A, B, C, D, E, F). For any such multiset, let k be the total number of elements of types A, B, or C that it has. These elements form a multiset of size k with elements of 3 types: these are counted in part (b). The remaining $n-k$ elements of the multiset have types D, E, or F: these form a multiset of size $n-k$ with elements of 3 types. As in part (b), there are

$$\binom{n-k+3-1}{3-1} = \binom{n-k+2}{2}$$

such multisets. Since k can be any number from 0 up to n, this accounts for all possibilit on the right-hand side of the formula.

Both sides of the formula count multisets of size n with elements of 6 types -- so the two sides of the formula are equal.

6. [6 pts.] A standard deck of 52 cards has 16 *face cards* (ranks J, Q, K, or A) and 36 *minor cards* (ranks 2 to 10).

There are $\binom{52}{4} = 270725$ ways to choose a hand of four cards from a standard deck of 52 cards. For each $0 \le k \le 4$, let $p(k)$ be the probability of getting **exactly** k face cards in a random hand of four cards. Complete the remaining entries in the following table:

# of face cards k	formula		# of outcomes	probability $p(k)$
0	$\binom{16}{0}\binom{36}{4}$	=	58905	0.217582...
1	$\binom{16}{1}\binom{36}{3}$	=	114240	0.421978...
2	$\binom{16}{2}\binom{36}{2}$	=	75600	0.279250...
3	$\binom{16}{3}\binom{36}{1}$	=	20160	0.074467...
4	$\binom{16}{4}\binom{36}{0}$	=	1820	0.006727...