CO453: Network Design – Winter 2007

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Assignment 6

Due: Monday, April 2, 2007 after class

You must give a proof of correctness of any algorithm you design, and argue briefly why it runs in polynomial time. You may use any proof or algorithm covered in class directly.

Q1: Consider the minimum spanning tree (MST) problem in a setting where rational players hold the edge costs. We are given a graph $G = (V, E)$, which is public information, known to everyone. Instead of assuming that every edge is a separate player, we consider the more general setting where there are k players and each player i owns a set $S_i \subseteq E$ of edges, where the sets S_i partition E (i.e., $\bigcup_{i=1}^{k} S_i = E$, $S_i \cap S_{i'} = \emptyset$). The sets S_i are also publicly known. The information that is private to a player i are the costs of the edges in his set S_i, which we will assume are non-negative. Given a spanning tree T, the cost incurred by each player i is the total cost of the edges in $S_i \cap T$. We would like to compute an MST with respect to the players' true edge costs in this setting.

(a) Formalize the problem as a mechanism design problem. That is, specify what is the set A of alternatives, for each i, the domain $V_i \subseteq \mathbb{R}^A$ of player i's valuation functions (representing costs incurred), and the target function $g : V \mapsto A$, where $V = V_1 \times \ldots \times V_k$, that needs to be computed. **(5 marks)**

(b) Assume that the graph G is such that the graph $G_i = (V, E \setminus S_i)$ is connected for every i. Recognize that the target function g is of the form for which the VCG theorem applies. Hence, give payment functions $p_i : V \mapsto \mathbb{R}$ for each player $i = 1, \ldots, k$ such that the mechanism $M = \big(g, \{p_i\}\big)$ is a truthful mechanism. Your payments should satisfy the property that the utility of every player is non-negative if he declares his true value. **(5 marks)**

Q2: Consider the set cover problem in a mechanism-design setting. We are given a universe U of n elements and a collection $\mathcal{S}$ of m subsets of U. The set-system $(U, \mathcal{S})$ is public knowledge. Each set S is a player, and his private value is the weight of the set (which we assume is non-negative), which is the cost he incurs if the set S is chosen in the set cover. So here,

$$A = \{\mathcal{S}' \subseteq \mathcal{S} : \text{ the sets in } \mathcal{S}' \text{ cover } U\}, \quad \text{and for each player (i.e., set) } S,$$

$$V_S = \{w_S : A \mapsto \mathbb{R} \text{ s.t. } w_S(\mathcal{S}') = 0 \text{ if } S \notin \mathcal{S}' \text{ and equal to some common value, say } w_S, \text{ otherwise}\}.$$

As usual, we are overloading notation and using w_S to denote both the functional mapping from A to $\mathbb{R}$ for a player S, and the common value assigned by this function, which represents weight of the set S, to any set-cover containing S. We would like to compute the minimum-weight set cover with respect to the players' true weights. That is, our target function is $g : V \mapsto A$, where $g(w) = \mathcal{S}'$ such that $\sum_{S \in \mathcal{S}'} w_S = \min_{\mathcal{S}'' \in A} \sum_{S \in \mathcal{S}''} w_S$.

(a) The target function g is of the form for which the VCG result applies. However since the function g cannot be efficiently computed, the VCG-mechanism does not give an efficient truthful mechanism. Explain briefly why simply replacing the function g by an approximation algorithm for the set cover problem and then computing payments as in the VCG mechanism, need not give a truthful mechanism. **(5 marks)**

(b) Notice that the players' domains V_S are *single-dimensional*. Thus, as shown in class, any function $f : V \mapsto A$ satisfying the following *monotonicity property* can be truthfully implemented:

for every player S, every w_{-S}, and values $w_S^1 > w_S^2$, if $S \in f(w_S^1, w_{-S})$ then $S \in f(w_S^2, w_{-S})$.

One of the approximation algorithms we gave for set cover was the following: obtain an optimal solution x^* to the LP-relaxation of the problem and simply pick every set S for which $x_S^* > 0$. Let f denote this approximation algorithm. Prove that f satisfies the above-stated monotonicity property. **(10 marks)**

(Hint: It suffices to show that if x^1 and x^2 optimal solutions to inputs (w_S^1, w_{-S}) and (w_S^2, w_{-S}), then $x_S^2 \geq x_S^1$; compare the values of x^1 and x^2, which are feasible fractional set covers, under the inputs (w_S^1, w_{-S}) and (w_S^2, w_{-S}).)