Degrees that are not degrees of categoricity

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Abstract

A computable structure $\mathcal{A}$ is $\mathbf{x}$-*computably categorical* for some Turing degree $\mathbf{x}$, if for every computable structure $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism $f : \mathcal{B} \to \mathcal{A}$ with $f \leq_T \mathbf{x}$. A degree $\mathbf{x}$ is a *degree of categoricity* if there is a computable structure $\mathcal{A}$ such that $\mathcal{A}$ is $\mathbf{x}$-computably categorical, and for all $\mathbf{y}$, if $\mathcal{A}$ is $\mathbf{y}$-computably categorical then $\mathbf{x} \leq_T \mathbf{y}$.

We construct a Σ^0_2 set whose degree is not a degree of categoricity. We also demonstrate a large class of degrees that are not degrees of categoricity by showing that every degree of a set which is 2-generic relative to some perfect tree is not a degree of categoricity. Finally, we prove that every noncomputable hyperimmune-free degree is not a degree of categoricity.

1 Introduction

Classically, isomorphic structures are considered to be equivalent. In computable structure theory, one has to be more careful. Different copies of the same structure may have different complexity, and for some structures, it can happen that there are two computable copies of the structure between which there is no computable isomorphism. In fact, for situations where this does not happen, we have the following definition.

Definition 1. A computable structure $\mathcal{A}$ is *computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$ there exists a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

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For example, any two computable dense linear orders without endpoints are computably isomorphic. Thus, any computable dense linear order without endpoints is computably categorical.

On the other hand, it is well known that the structure $(\mathbb{N}, <)$, the natural numbers with the usual $<$ order, is not computably categorical. Indeed, let $\{K_s\}_{s\in\omega}$ be a computable enumeration of $\emptyset'$ where there is exactly one element enumerated at each stage, and consider the order $\mathcal{A}$ where the even numbers have their usual order and $2n <_{\mathcal{A}} 2s+1 <_{\mathcal{A}} 2n+2$ iff $n \in K_{s+1} - K_s$. Note that any isomorphism $f : \mathcal{A} \to \mathcal{N}$ computes $\emptyset'$. Conversely, between any two computable copies of $(\mathbb{N}, <)$ there is a $\emptyset'$-computable isomorphism. That is, it seems that $\mathbf{0}'$ is the degree of difficulty associated to the problem of computing isomorphisms between arbitrary copies of $(\mathbb{N}, <)$. This motivates the following definitions.

Definition 2. A computable structure $\mathcal{A}$ is **d**-*computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$ there exists a **d**-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

So with this definition, $(\mathbb{N}, <)$ is $\mathbf{0}'$-computably categorical.

Definition 3. We say a structure $\mathcal{A}$ has *degree of categoricity* **d** if $\mathcal{A}$ is **d**-computably categorical, and for all **c** such that $\mathcal{A}$ is **c**-computably categorical, $\mathbf{d} \le \mathbf{c}$. We say a degree **d** is a *degree of categoricity* if there is some structure with degree of categoricity **d**.

So in our examples, we have seen that $\mathbf{0}$ and $\mathbf{0}'$ are degrees of categoricity.

The notion of a degree of categoricity was first introduced by Fokina, Kalimullin and Miller in [3]. In that paper, they showed that if **d** is d.c.e. in and above $\mathbf{0}^{(\mathbf{n})}$, then **d** is a degree of categoricity. They also showed that $\mathbf{0}^{(\omega)}$ is a degree of categoricity. In fact, all the examples they constructed had the following, stronger property.

Definition 4. A degree of categoricity **d** is a *strong* degree of categoricity if there is a structure $\mathcal{A}$ with computable copies $\mathcal{A}_0$ and $\mathcal{A}_1$ such that **d** is the degree of categoricity for $\mathcal{A}$, and every isomorphism $f : \mathcal{A}_0 \to \mathcal{A}_1$ satisfies $\deg(f) \ge \mathbf{d}$.

Fokina, Kalimullin and Miller [3] showed that all strong degrees of categoricity are hyperarithmetical. Later, Csima, Franklin and Shore [2] showed that in fact all degrees of categoricity are hyperarithmetical. This may or may not be an improvement, as it is unknown whether all degrees of categoricity are strong.

Csima, Franklin and Shore [2] have shown that for every computable ordinal α, $\mathbf{0}^{(\alpha)}$ is a (strong) degree of categoricity. They also showed that if α is a computable successor ordinal and **d** is d.c.e. in and above $\mathbf{0}^{(\alpha)}$, then **d** is a (strong) degree of categoricity.

The work on degrees of categoricity so far has gone into showing that various degrees *are* degrees of categoricity. In this paper we address the question: What are examples of degrees that are *not* degrees of categoricity? Certainly, as there are only countably many computable structures, there are only countably many degrees of categoricity. In section 3, we give a basic construction of a degree below $\mathbf{0}''$ that is not a degree of categoricity. In section 4 we show that degrees of 2-generics (indeed, of 2-generics relative to perfect trees) are not degrees of categoricity. In section 5 we show that noncomputable hyperimmune-free degrees cannot be degrees of categoricity. Finally, in section 6, we show that there exists a Σ^0_2 degree that is not a degree of categoricity.

2 Notation

For general references, see Harizanov [4] for computable structure theory, and Soare [7] for computability theory.

We use T to denote a tree (a subset of $2^{<\omega}$ closed under initial segments). All other uppercase letters are used for subsets of the natural numbers and lowercase bold letters are used for Turing degrees. We use α, β, σ, and τ to represent strings (elements of $2^{<\omega}$). When dealing with strings, $\supseteq$ denotes string extension. For $A \subseteq \mathbb{N}$ and $n \in \omega$, we let $A \upharpoonright n = \{x \in A \mid x < n\}$ and let $A \upharpoonright\upharpoonright n = \{x \in A \mid x \leq n\}$, with analogous definitions of $\sigma \upharpoonright n$ and $\sigma \upharpoonright\upharpoonright n$ for $\sigma \in 2^{<\omega}$. We have Φ_n denote the n-th oracle Turing reduction and φ_n the n-th Turing reduction.

We use calligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{M})$ to denote computable structures. We let $\mathcal{A}_n$ denote the n-th partial computable structure under some effective listing. For simplicity, we assume all computable structures have domain ω or an initial segment of ω. We also assume all structures are in a finite language.

We let $\mathrm{Part}(\mathcal{A}, \mathcal{B})$ denote the set of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. That is, the set of functions that, on their domain/range, are injective homomorphisms from a substructure of $\mathcal{A}$ to a substructure of $\mathcal{B}$. Note that if $\mathcal{A}_i$ and $\mathcal{A}_j$ are computable structures according to our listing, then if for some σ and some t we have $\Phi^{\sigma}_{e,t} \notin \mathrm{Part}(\mathcal{A}_{i,t}, \mathcal{A}_{j,t})$, then for all $A \supset \sigma$, $\Phi^A_e \notin \mathrm{Part}(\mathcal{A}_i, \mathcal{A}_j)$. Also, if f is a bijection such that for all n, $f \upharpoonright n \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$, then $f : \mathcal{A} \cong \mathcal{B}$.

For $\sigma, \tau \in 2^{<\omega}$ we write $\sigma <_L \tau$ if $\sigma \subset \tau$ or if there is some n such that $\sigma(n) = 0$, $\tau(n) = 1$, and for all $k < n$, $\sigma(k) = \tau(k)$. That is, if σ comes before τ in the usual lexicographical order. We say a set A is left-c.e. if there is a computable sequence $\alpha_s \in 2^{<\omega}$, where $A(n) = \lim_s \alpha_s(n)$, and for all s, $\alpha_s <_L \alpha_{s+1}$. Left c.e. sets have been studied extensively in the area of algorithmic randomness (see Nies [6]). It is easy to see that all c.e. sets are also left-c.e. The converse does not hold. However, every left-c.e. set is Turing equivalent to a c.e. set (since $A \equiv_T \{\sigma \mid \sigma <_L A\}$), and it is this feature of left-c.e. sets that we will make use of.

Finally, we use the following definition.

Definition 5. Let $\mathcal{A}$ be a computable structure. We define $\mathrm{CatSpec}(\mathcal{A})$ to be the set of degrees $\mathbf{d}$ such that $\mathcal{A}$ is $\mathbf{d}$-computably categorical.

3 Basic construction

It will follow from several of the results in this paper that there is a degree $\mathbf{x} \leq_T \mathbf{0}''$ which is not a degree of categoricity. However we will briefly sketch a proof of this fact here, since the ideas we use are expanded on in the proofs in sections 4 and 6.

For the proof, we will construct a noncomputable set X such that for all m, k either X does not compute an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, or there is a computable isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$. Given the construction, suppose the degree $\mathbf{x}$ of X is a degree of categoricity, witnessed by $\mathcal{A}$. Let $\mathcal{B}$ be an arbitrary computable copy of $\mathcal{A}$. Since $\mathcal{A}$ is $\mathbf{x}$-computably categorical, X computes an isomorphism from $\mathcal{B}$ to $\mathcal{A}$. By the construction, there is then a computable isomorphism from $\mathcal{B}$ to $\mathcal{A}$. Since $\mathcal{B}$ was arbitrary, $\mathcal{A}$ is computably categorical, for a contradiction.

We will build X by finite extensions using a $\emptyset''$ oracle. At each stage we will use the $\emptyset''$ oracle to try to extend X to block some Φ_l^X from being an isomorphism. If such a block is not possible, we will argue that a computable isomorphism can be found.

Proposition 3.1. *There is a degree $\mathbf{x} \leq_T \mathbf{0}''$ such that $\mathbf{x}$ is not a degree of categoricity.*

Proof (sketch). We build X by finite extensions using a $\emptyset''$ oracle. We start with $X_0 = \langle\rangle$. For stage $s+1$, we will ensure that either Φ_l^X is not an isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$ or there is a computable isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$, where $s = \langle l, m, k\rangle$.

We start stage $s+1$ by using $\emptyset'$ to diagonalize against X being computable by φ_s. We then use $\emptyset'$ to determine if there is a $\sigma \supseteq X_s$ and a time t such that $\Phi_{l,t}^\sigma$ can be seen not to be an injective homomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$. If there is, we let $X_{s+1} = \sigma$ and proceed to the next stage. If there is not, we ask $\emptyset''$ if there exists $\sigma \supseteq X_s$ and $n \in \omega$ such that for all $\tau \supseteq \sigma$ we have Φ_l^τ omits n from its domain or range. If such a σ exists, we note Φ_l^Y is not an isomorphism for any $Y \supseteq \sigma$, so we let $X_{s+1} = \sigma$.

If the answer to both questions is no, then for any $\gamma \supseteq X_s$ we have that Φ_l^γ is a partial injective homomorphism and for every n there is a $\tau \supseteq \gamma$ with n in the domain and range of Φ_l^τ. Note this τ also extends X_s, so these properties also hold for τ. We let $\alpha_0 = X_s$ and α_{n+1} be the first extension of α_n which puts n into the domain and range of $\Phi_l^{\alpha_{n+1}}$. Letting $A = \bigcup_{n\in\omega} \alpha_n$ and $f = \Phi_l^A$, we have that f is a computable isomorphism from $\mathcal{A}_m$ to $\mathcal{A}_k$. Thus we let $X_{s+1} = X_s$ and move to the next stage.

This completes our construction of X. We have $X \leq_T \emptyset''$, and as noted in the explanation before the proof, the degree $\mathbf{x}$ of X is not a degree of categoricity. $\square$

4 2-generic relative to some perfect tree

We wish to generalize Proposition 3.1 to show that a large class of sets have degrees that are not degrees of categoricity. To do this we will use the concept of sets that are n-generic relative to some perfect tree. Recall a set G is n-generic if for every Σ_n^0 set, either it meets the set or some initial segment cannot be extended to meet the set.

Definition 6. A set G is n-generic if for every Σ_n^0 subset S of $2^{<\omega}$, either there is an l such that $G \upharpoonright l \in S$, or there is an l such that for all $\sigma \supseteq G \upharpoonright l$ we have $\sigma \notin S$.

We relativize this notion from $2^{<\omega}$ to a perfect tree.

Definition 7. A set G is n-generic relative to the perfect tree T if G is a path through T and for every $\Sigma_n^0(T)$ subset S of $2^{<\omega}$, either there is an l such that $G \upharpoonright l \in S$, or there is an l such that for all $\sigma \supseteq G \upharpoonright l$ with $\sigma \in T$ we have $\sigma \notin S$.

Definition 8. A set G is n-generic relative to some perfect tree if there exists a perfect tree T such that G is n-generic relative to T.

It has been shown that almost all sets are 2-generic relative to some perfect tree.

Theorem 4.1 (Anderson [1])**.** *For any n, all but countably many sets are n-generic relative to some perfect tree.*

We will prove that every degree containing a set that is 2-generic relative to some perfect tree is not a degree of categoricity. As a result we are able to limit the degrees of categoricity to an easily defined countable class (distinct from HYP). It will follow as a corollary that for any degree $\mathbf{x}$ there is a degree $\mathbf{y}$ with $\mathbf{x} \leq_T \mathbf{y} \leq_T \mathbf{x}''$ such that $\mathbf{y}$ is not a degree of categoricity.

We can view the proof of our theorem as the proof to Proposition 3.1 relativized twice, in successive stages. We first relativize the proof from a single Δ_3^0 set to any 2-generic set. The idea is that if G computes an isomorphism then we can find an initial segment $G \upharpoonright l$ such that for every extension of $G \upharpoonright l$ the answer to both of the questions we ask in the original proof is no. We can then build a computable isomorphism as in the original proof.

We next relativize from every 2-generic to every 2-generic relative to an arbitrary perfect tree T. The key here is that the proof of Proposition 3.1 is stronger than required. In the original proof, we show that if G computes an isomorphism then there is a computable one. It suffices to fix some $H \not\geq_T G$ such that if G computes an isomorphism then so does H. When we relativize to T we obtain this for T in the place of H.

Theorem 4.2. *Let G be 2-generic relative to some perfect tree. Then the degree of G is not a degree of categoricity.*

Proof. Let G be 2-generic relative to the perfect tree T. Suppose the degree of G is a degree of categoricity, witnessed by $\mathcal{A}$. Let $\mathcal{B}$ be an arbitrary computable structure such that $\mathcal{A}$ is isomorphic to $\mathcal{B}$. We will show there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ with $f \leq_T T$. Since our choice of $\mathcal{B}$ is arbitrary, we can then conclude $T \in \mathrm{CatSpec}(\mathcal{A})$, so $T \geq_T G$ for a contradiction.

Let Ψ be such that Ψ^G is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Let R_s be the set of strings σ such that Ψ_s^σ contains values contradicting it being a partial injective homomorphism from $\mathcal{A}$ to $\mathcal{B}$, i.e. $\Psi_s^\sigma \notin \mathrm{Part}(\mathcal{A}, \mathcal{B})$. We note R_s is (uniformly) computable. Let

$$S = \{\sigma \in 2^{<\omega} \mid \exists n \, \forall s \, \forall \tau \in T[\tau \supseteq \sigma \to \\ (\tau \in R_s \ \vee \ n \notin \mathrm{dom}(\Psi_s^\tau) \ \vee \ n \notin \mathrm{ran}(\Psi_s^\tau))]\}$$

We note S is $\Sigma_2^0(T)$. Suppose for some j we have $G \upharpoonright j \in S$, witnessed by n. Since Ψ^G is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, let $m > j$ and s be large enough so that n is in the domain and range of $\Psi_s^{G \upharpoonright m}$. Then letting τ be $G \upharpoonright m$ we have $G \upharpoonright m \in R_s$, contradicting Ψ^G being an isomorphism. We conclude that G does not meet S.

By genericity, there is an l such that for all $\sigma \in T$ with $\sigma \supseteq G \upharpoonright l$ we have $\sigma \notin S$. Hence we have:

$$\forall \sigma \in T \, [\sigma \supseteq G \upharpoonright l \to \forall n \, \exists s \, \exists \tau \in T \\ [\tau \supseteq \sigma \ \wedge \ \tau \notin R_s \ \wedge \ n \in \mathrm{dom}(\Psi_s^\tau) \ \wedge \ n \in \mathrm{ran}(\Psi_s^\tau)]] \tag{1}$$

We can now construct our isomorphism $f : \mathcal{A} \to \mathcal{B}$ with $f \leq_T T$ to complete the proof. We will T-computably build $A = \bigcup_{i \in \omega} \alpha_i$ so that $f = \Psi^A$.

Let α_0 be $G \upharpoonright l$. Given α_i, let α_{i+1} be the first τ we find satisfying (1) with i for n and α_i for σ. We note that every $\alpha_i \supseteq G \upharpoonright l$ (and $\alpha_i \in T$), so finding a τ which satisfies (1) is always possible. This completes our construction.

We note $A \leq_T T$ so $f \leq_T T$. From the construction it is clear that f is total and surjective. To show that f is an isomorphism, it suffices to show that for all i and t we have $\alpha_i \notin R_t$.

Suppose $\alpha_i \in R_t$ for some i, t. Let $j > i$ be sufficiently large such that $j \in \mathrm{dom}(\Psi_s^A)$ requires $s > t$. We then have $\alpha_j \notin R_s$ for some $s > t$ so $\alpha_j \notin R_t$. Since $\alpha_i \subseteq \alpha_j$ we have $\alpha_i \notin R_t$ for a contradiction. We conclude that for all i, t, we have $\alpha_i \notin R_t$. Thus f is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

As noted at the start of the proof, this implies $G \leq_T T$ for a contradiction. We conclude G is not a degree of categoricity. □

Corollary 4.3. *Let A be a set and let G be 2-generic(A). Then the degree of $G \oplus A$ is not a degree of categoricity.*

Proof. Let $T = \{\sigma \in 2^{<\omega} \mid \exists \tau \in 2^{<\omega} \, [\sigma \subseteq \tau \oplus A]\}$. Then $G \oplus A$ is 2-generic relative to T so by Theorem 4.2, the degree of $G \oplus A$ is not a degree of categoricity. □

Corollary 4.4. *Let* $\mathbf{x}$ *be any Turing degree. Then there exists* $\mathbf{y}$ *with* $\mathbf{x} \leq_T \mathbf{y} \leq_T \mathbf{x}''$ *such that* $\mathbf{y}$ *is not a degree of categoricity.*

Proof. Let X be a set of degree $\mathbf{x}$ and let $G \leq_T X''$ be 2-generic(X). Let $\mathbf{y}$ be the degree of $X \oplus G$. Then $\mathbf{x} \leq_T \mathbf{y} \leq_T \mathbf{x}''$ and by Corollary 4.3, $\mathbf{y}$ is not a degree of categoricity. □

5 Hyperimmune-free

Recall that a degree $\mathbf{y}$ is hyperimmune-free if every function $f \leq_T \mathbf{y}$ can be bounded by a computable function (see [7]). We note that all known degrees of categoricity $\mathbf{x}$ are such that $\mathbf{0}^{(\gamma)} \leq_T \mathbf{x} \leq_T \mathbf{0}^{(\gamma+1)}$ for some ordinal γ, and hence are hyperimmune (or computable). This suggests the question, is there a (noncomputable) degree of categoricity which is hyperimmune-free? We show that no such degree exists.

Theorem 5.1. *Let* $\mathbf{b}$ *be a noncomputable hyperimmune-free degree. Then* $\mathbf{b}$ *is not a degree of categoricity.*

Proof. Let $\mathbf{b}$ be a noncomputable hyperimmune-free degree, and assume for a contradiction that $\mathbf{b}$ is a degree of categoricity. Let $\mathcal{A}$ witness that $\mathbf{b}$ is a degree of categoricity. Let $\mathcal{B}$ be an arbitrary computable structure such that $\mathcal{A}$ is isomorphic to $\mathcal{B}$. We will show there is an isomorphism $g : \mathcal{A} \to \mathcal{B}$ such that $g \leq_T \emptyset'$. Since $\mathcal{B}$ is arbitrary, we will then have $\mathbf{0}' \in \mathrm{CatSpec}(\mathcal{A})$. Hence $\mathbf{b} \leq \mathbf{0}'$, contradicting $\mathbf{b}$ being noncomputable and hyperimmune-free. Therefore it suffices to show there exists such a g.

Let $f : \omega \to \omega$ be an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ with $f \leq_T \mathbf{b}$. We note since f is bijective, $f^{-1} \leq_T \mathbf{b}$. Since $\mathbf{b}$ is hyperimmune-free, let h be a computable function which dominates f and f^{-1}.

We now use h to build an infinite computably bounded tree $T \subset \omega^{<\omega}$ whose infinite paths code isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. Then $[T]$ must have a $\emptyset'$-computable member (indeed, a low member), so there exists $g \leq_T \emptyset'$ with $g : \mathcal{A} \cong \mathcal{B}$ as desired.

The infinite paths through T will code isomorphisms by having the map from $\mathcal{A}$ to $\mathcal{B}$ on the even bits, and the inverse map on the odd bits. For $\sigma \in \omega^{<\omega}$, let $\sigma_0(n) = \sigma(2n)$ and $\sigma_1(n) = \sigma(2n+1)$. Let T be defined by:

$$T = \{\sigma \in \omega^{<\omega} \mid \forall n \leq \mathrm{length}(\sigma)[[\sigma(n) \leq h(\lfloor \frac{n}{2} \rfloor)] \wedge \sigma_0 \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \wedge \\ \sigma_1 \in \mathrm{Part}(\mathcal{B}, \mathcal{A}) \wedge (i \neq j \to (\sigma_i(\sigma_j(n)) = n \vee \sigma_i(\sigma_j(n)) \uparrow))]\}$$

Then T is a computably bounded tree. Let $\tilde{f}(2n) = f(n)$ and $\tilde{f}(2n+1) = f^{-1}(n)$. Then $\tilde{f} \in [T]$, so T is infinite. Let $\tilde{g} \in T$ be such that $\tilde{g} \leq_T \emptyset'$. Let $g(n) = \tilde{g}(2n)$. Then $g \leq_T \tilde{g} \leq_T \emptyset'$ and $g : \mathcal{A} \cong \mathcal{B}$ as desired. □

6 Σ^0_2 degree

Theorem 6.1. *There is a* Σ^0_2 *degree that is not a degree of categoricity.*

Proof. We build a set D with Σ^0_2-degree $\mathbf{d}$ that is not a degree of categoricity. This time, instead of a $\emptyset''$-oracle construction, we build D to be left-c.e. in $\emptyset'$. We again meet the requirements:

$R_{\langle e,i,j\rangle}$: If $\Phi_e^D : \mathcal{A}_i \cong \mathcal{A}_j$ then there is a computable isomorphism between $\mathcal{A}_i$ and $\mathcal{A}_j$.

If, for example, $R_{\langle e,i,j\rangle}$ were the highest priority requirement, we would ask $\emptyset'$, $(\exists\sigma \supseteq 1)(\exists t)(\Phi^\sigma_{e,t} \notin \mathrm{Part}(\mathcal{A}_{i,t}, \mathcal{A}_{j,t})$? If yes, we would extend to σ.

If no, then (while letting $\delta_0 = 0$ and also addressing other requirements) at all subsequent stages s we would ask $\emptyset'$, $(\forall\sigma \supseteq 1, |\sigma| = s)(\forall k \leq s)(\exists\tau \supseteq \sigma)(\exists t)(k \in \mathrm{dom}\Phi^\tau_{e,t} \wedge k \in \mathrm{rng}\Phi^\tau_{e,t})$?

If the answer is always "yes", then we can use Φ_e to build a computable isomorphism between $\mathcal{A}_i$ and $\mathcal{A}_j$.

If the answer is "no" at some stage, then at that stage we set $\delta_s = \sigma$ such that Φ_e^σ cannot be extended to an isomorphism. This is a move that is left-c.e. in $\emptyset'$, and causes injury to lower priority requirements.

We now give the formal construction. Of course, in addition to the $R_{\langle e,i,j\rangle}$ requirements discussed above, we must also meet non-computability requirements:

$N_e : D \neq \varphi_e$.

At each stage s, requirements of the form $R_{\langle e,i,j\rangle}$ will either be *unsatisfied*, *under consideration*, or *satisfied*. Requirements of the form N_e will either be *unsatisfied* or *satisfied*. The status of each requirement will change at most finitely often. We will have δ_s denote the stage s approximation to D. We will either have $\delta_{s+1} \supset \delta_s$, or $\delta_{s+1} \supseteq \sigma_{\langle e,i,j\rangle}$ where $\delta_s \supseteq \delta_k\widehat{\ }0$ and $\sigma_{\langle e,i,j\rangle} \supseteq \delta_k\widehat{\ }1$ for some $k < s$, so that the approximation will be left-c.e. in $\emptyset'$.

Stage 0: Set $\delta_0 = \emptyset$.

Stage s+1=2m +1: For each $R_{\langle e,i,j\rangle}$ currently under consideration, in turn, ask $\emptyset'$, $(\forall\sigma \supseteq \sigma_{\langle e,i,j\rangle}, |\sigma| = s)(\forall k \leq s)(\exists\tau \supseteq \sigma)(\exists t)(k \in \mathrm{dom}\Phi^\tau_{e,t} \wedge k \in \mathrm{rng}\Phi^\tau_{e,t})$? If there is a least $R_{\langle e,i,j\rangle}$ for which the answer is "no", then let $\delta_{s+1} \supseteq \sigma_{\langle e,i,j\rangle}$ be such that $(\exists k)(\forall\tau \supseteq \delta_{s+1})(k \notin \mathrm{dom}\Phi^\tau_e \vee k \notin \mathrm{rng}\Phi^\tau_e)$. Declare $R_{\langle e,i,j\rangle}$ to be satisfied. Declare N_n and R_n to be unsatisfied for all $n > \langle e,i,j\rangle$, canceling any associated σ_n for those R_n that were under consideration.

If the answer was always "yes", let $\langle e,i,j\rangle$ be least such that $R_{\langle e,i,j\rangle}$ is unsatisfied. Ask $\emptyset'$, $(\exists\sigma \supseteq \delta_s\widehat{\ }1)(\exists t)(\Phi^\sigma_{e,t} \notin \mathrm{Part}(\mathcal{A}_{i,t}, \mathcal{A}_{j,t})$? If the answer is "yes", let δ_{s+1} have this property, and declare $R_{\langle e,i,j\rangle}$ to be satisfied. If the answer is "no", let $\sigma_{\langle e,i,j\rangle} = \delta_s\widehat{\ }1$, let $\delta_{s+1} = \delta_s\widehat{\ }0$, and declare $R_{\langle e,i,j\rangle}$ to be under consideration.

Stage s+1=2m +2: Let n be least such that N_n is not satisfied. Use $\emptyset'$ to determine whether $(\exists t)(\varphi_{n,t}(|\delta_s|) = 0)$. If yes, define $\delta_{s+1} = \delta_s\widehat{\ }1$. Otherwise, define $\delta_{s+1} = \delta_s\widehat{\ }0$. Declare N_n to be satisfied.

This completes the construction.

Lemma 6.2. *The approximation $\{\delta_s\}_{s\in\omega}$ defines a set D that is left-c.e. in $\emptyset'$.*

Proof. The construction is $\emptyset'$-computable, so the sequence $\{\delta_s\}_{s\in\omega}$ is $\emptyset'$-computable. We either have $\delta_{s+1} \supset \delta_s$, or we have $\delta_{s+1} \supseteq \sigma_{\langle e,i,k\rangle}$ for some $R_{\langle e,i,k\rangle}$ that was under consideration at stage $s+1$. In the first case, certainly $\delta_s <_L \delta_{s+1}$. In the second case, note that at the greatest stage $t+1 \leq s$ where $\sigma_{\langle e,i,j\rangle}$ was defined, we had let $\delta_{t+1} = \delta_t \widehat{\ } 0$ and $\sigma_{\langle e,i,j\rangle} = \delta_t \widehat{\ } 1$. As $R_{\langle e,i,j\rangle}$ remained under consideration between stages $t+1$ and $s+1$, there was no shift to σ_k for any $k \leq \langle e,i,j\rangle$ between stages $t+1$ and s. It is easy to see by induction on $t+1 \leq t' \leq s$ that $\delta_{t'} \supseteq \delta_{t+1}$ and that if σ_n was defined at stage t' then $\sigma_n \supseteq \delta_{t'}$. So $\delta_s \supseteq \delta_t \widehat{\ } 0$ and $\delta_{s+1} \supseteq \delta_t \widehat{\ } 1$. That is, the approximation $\{\delta_s\}_{s\in\omega}$ is left-c.e. in $\emptyset'$. □

Lemma 6.3. *For each n, there is a stage s after which the status of requirement R_n ceases to change. Each requirement R_n is met.*

Proof. Consider the requirement $R_{\langle e,i,j\rangle}$, and let s be the least stage by which the status of all R_n for $n < \langle e,i,j\rangle$ cease to change. Note that $R_{\langle e,i,j\rangle}$ had status "unsatisfied" at stage s. At stage $s+1$, the status of $R_{\langle e,i,j\rangle}$ becomes "satisfied" or "under consideration". The only way for the status of $R_{\langle e,i,j\rangle}$ to change back to unsatisfied would be if there was a change in status for some R_n with $n < \langle e,i,j\rangle$. So $R_{\langle e,i,j\rangle}$ is never again unsatisfied.

Suppose there is a stage greater than s where $R_{\langle e,i,j\rangle}$ becomes satisfied. Let t be the least such stage. At stage t we set δ_t such that $\Phi_e^{\delta_t} \notin \mathrm{Part}(\mathcal{A}_{i,t}, \mathcal{A}_{j,t})$, or such that there exists k such that for all $\tau \supseteq \gamma_t$, $k \notin \mathrm{dom}\Phi_e^\tau$ or $k \notin \mathrm{rng}\Phi_e^\tau$. Since there is no change in status for any R_n with $n < \langle e,i,j\rangle$ beyond stage s, it is easy to see by induction on $t' > t$ that $\delta_{t'} \supseteq \delta_{t+1}$ and that if σ_n was defined at stage t' then $\sigma_n \supseteq \delta_{t'}$. So $D \supset \delta_t$ and $\Phi_e^D \notin \mathrm{Part}(\mathcal{A}_i, \mathcal{A}_j)$, i.e., $R_{\langle e,i,j\rangle}$ is met, and indeed has the status satisfied at all stages beyond t.

Suppose there is no stage greater than s where $R_{\langle e,i,j\rangle}$ becomes satisfied. In this case, $R_{\langle e,i,j\rangle}$ maintains status under consideration at all stages greater than s, and $\sigma_{\langle e,i,j\rangle}$ does not change after it is defined at stage $s+1$. We now show that $R_{\langle e,i,j\rangle}$ is met, by building a computable isomorphism $f : \mathcal{A}_i \to \mathcal{A}_j$. First note that $(\forall \sigma \supset \sigma_{\langle e,i,j\rangle})(\forall t)(\Phi_{e,t}^\sigma \in \mathrm{Part}(\mathcal{A}_{i,t}, \mathcal{A}_{j,t}))$. Since $R_{\langle e,i,j\rangle}$ remains under consideration at all stages $n > s$, $(\forall \sigma \supseteq \sigma_{\langle e,i,j\rangle}, |\sigma| = n)(\forall k \leq n)(\exists \tau \supseteq \sigma)(\exists t)(k \in \mathrm{dom}\Phi_{e,t}^\tau \wedge k \in \mathrm{rng}\Phi_{e,t}^\tau)$. Let $\tau_0 = \sigma_{\langle e,i,j\rangle}$. Given τ_n, let $\tau_{n+1} \supset \tau_n$ be such that $n \in \mathrm{dom}\Phi_e^\tau$ and $n \in \mathrm{rng}\Phi_e^\tau$, and $|\tau_{n+1}| \geq n+1$. Since we know such τ_{n+1} exists, we can search and find such effectively. Let $B = \cup_{n\in\omega}\tau_n$. Then B is computable, and $\Phi_e^B : \mathcal{A}_i \cong \mathcal{A}_j$.

□

Lemma 6.4. *The requirements N_n are all met.*

Proof. We prove by induction on n that if s is the least stage when all requirements R_i and N_j for $i \leq n$, $j < n$ cease to change status, then N_n is satisfied at all stages beyond stage $s+1$. Indeed, suppose the result holds for $k < n$, and that R_n obtains its final status for the first time at stage s. At stage $s+1$, N_n is either already satisfied, or receives attention and becomes satisfied. Since all the R_k for $k \leq n$ do not change status after stage s, it is easy to see that $D \supset \delta_{s+1}$, so N_n remains satisfied beyond stage $s+1$. □

□

Remark 6.5. *The set D constructed in Theorem 6.1 is such that $D \not\geq_T \emptyset'$.*

Proof. Fokina, Kalimullin, and Miller [3] showed that all degrees which are c.e.a. ($\emptyset'$) are degrees of categoricity. If we had $D \geq_T \emptyset'$ then D would be c.e.a. ($\emptyset'$) for a contradiction. $\square$

We note that since all Σ_2^0 degrees are hyperimmune, this implies there is a hyperimmune degree which is not a degree of categoricity [5].

7 Conclusion

Considerable ground remains open in finding how low in complexity a degree can be without being a degree of categoricity. On one side, it is not known if there is a Δ_2^0 degree which is not a degree of categoricity. On the other side, it has not been shown that every 3-c.e. degree is a degree of categoricity.

We can also consider questions about categorizing the degrees of categoricity. What other classes of degrees can be shown to lack (or have) degrees of categoricity? Must every degree of categoricity $\mathbf{x}$ be such that $\mathbf{0}^{(\gamma)} \leq_T \mathbf{x} \leq_T \mathbf{0}^{(\gamma+1)}$ for some ordinal γ?

Finally, the connection between degrees of categoricity and strong degrees of categoricity can be further explored. The question of Fokina, Kalimullin, and Miller [3], is there a degree of categoricity which is not strong, remains open. In fact, every computable structure constructed so far that witnesses a degree of categoricity, does so by witnessing a strong degree of categoricity.

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