PUTNAM TRAINING PROBLEMS 2001.6
Joyful complexity

1. Show that a necessary and sufficient condition for three points a, b, c in the complex plane to form an equilateral triangle is that

$$a^2 + b^2 + c^2 = bc + ca + ab.$$

2. Show that the polynomial $p(z) = z^5 - 6z + 3$ has five distinct complex roots, of which *exactly three* are real.

3. Let a and b be nonzero complex numbers and $f(z) = az + bz^{-1}$. Determine the image under f of the unit circle $\{z : |z| = 1\}$.

4. Give an example of a continuous real-valued function on the interval $[0, 1]$ that has more than two continuous square roots on $[0, 1]$.

5. Determine the complex numbers z for which the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n^{\log n}}$$

and its term by term derivatives of all orders converge absolutely.

6. Let u be a positive function on $\mathbb{R}^2$ satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Show that u is constant.

7. Let f and g be entire functions such that $\lim_{z\to\infty} f(g(z)) = \infty$. Prove that f and g are polynomials.