PUTNAM TRAINING PROBLEMS 2001.5
Period pieces

1. Let $f : I\!R \to I\!R - \{3\}$ be a function with the property that for some $\omega > 0$
$$f(x+\omega) = \frac{f(x)-5}{f(x)-3},$$
for all real x. Prove that f is periodic.

2. If a function $f : I\!R \to I\!R$ is *not* injective and there exists a function $g : I\!R \times I\!R \to I\!R$ such that $f(x+y) = g(f(x), y)$ for all $x, y \in I\!R$, show that f is periodic.

3. Find all real values α with the property that a sequence satisfying the recurrence relation $x_{n+1} + x_{n-1} = \alpha\, x_n$, $n \in I\!N$, is periodic.

4. Let b_n be the last digit of the number
$$1^1 + 2^2 + 3^3 + \cdots + n^n.$$
Prove that the sequence $b_1, b_2, b_3, \ldots$ is periodic with period 100.

5. (Putnam 1955 B5.) Given an infinite sequence of 0's and 1's and a fixed integer k, suppose there are no more than k distinct blocks of k consecutive terms. Show that the sequence is eventually periodic.

6. (Putnam 1964 A4.) Let p_n, $n = 1, 2, \ldots$, be a bounded sequence of integers which satisfies the recursion
$$p_n = \frac{p_{n-1} + p_{n-2} + p_{n-3}p_{n-4}}{p_{n-1}p_{n-2} + p_{n-3} + p_{n-4}}.$$
Show that the sequence eventually becomes periodic.

7. Let $f : I\!R \to I\!R$ be a periodic function such that the set $\{f(n) : n \in I\!N\}$ has infinitely many elements. Prove that the period of f is irrational.

8. The positive integers $a_1, a_2, a_3, \ldots$, all less than or equal to 2001, satisfy the following condition: $a_m + a_n$ is divisible by a_{m+n} for all positive integers m, n. Prove that this sequence is eventually periodic.