PUTNAM TRAINING PROBLEMS 2000.5
Rooting for Answers to Algebraic Appetizers

1. Let $f(x) = x^3 - 3x + 1$, where x is real. find the number of distinct real roots of the equation $f(f(x)) = 0$.

2. Let $a_1, \ldots, a_n$ be real numbers , not all zero. Prove that the equation
$$\sqrt{1 + a_1\, x} + \cdots + \sqrt{1 + a_n\, x} = n$$
has at most one nonzero real root.

3. Find all real numbers x such that
$$x \lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor = 88.$$

4. Two positive integers are written on the board. The following operation is repeated: if $a < b$ are the numbers on the board, then a is erased and $ab/(b-a)$ is written in its place. At some point the numbers on the board are found to be equal. Prove that again they are positive integers.

5. The numbers 19 and 98 are written on a board. Each minute, each number is either incremented by 1 or squared. Is it possible for the numbers to become identical at some time?

6. A binary operation $*$ on real numbers has the property that $(a * b) * c = a + b + c$. Prove that $a * b = a + b$.

7. Let P be the set of all points in $\mathbb{R}^n$ with rational coordinates. For A, $B \in P$, one can move from A to B if the distance AB is 1. Prove that every point in P can be reached from every other point in P by a finite sequence of moves if and only if $n \geq 5$.