

PUTNAM TRAINING PROBLEMS 2000.4
Logical Lumps and Subtle Sets

1. A jury determines with certainty that
 - (a) If A is not guilty, then B and C are both guilty.
 - (b) Either A is not guilty or B is guilty.
 - (c) Either B is not guilty or C is not guilty.

Determine the guilt or innocence of A, B and C.

2. Three counters A, B and C are coloured red, white and blue but not necessarily respectively. Of the following statements, only one is true:

A is red; B is not red; C is not blue.

What colour is each counter?

3. What is the largest number of distinct sets that can be obtained from one set by repeated applications of closure and complementation?
4. Let X be a collection of distinct subsets of some countable set S with the property that the intersection of any two elements of X is finite. Prove or disprove: X is a countable set.
5. (Putnam 1964.) Let S be a set of $n > 0$ elements, and let A_1, \dots, A_k be a family of distinct subsets, with the property that any two of these subsets meet. Assume that no other subset of S meets all of the A_i . Prove that $k = 2^{n-1}$.
6. (Putnam 1980.) Let A_1, \dots, A_{1066} be subsets of a finite set X such that $2 \text{card}(A_i) > \text{card}(X)$ for $1 \leq i \leq 1066$. Prove there exist ten elements x_1, \dots, x_{10} of X such that every A_i contains at least one of x_1, \dots, x_{10} .