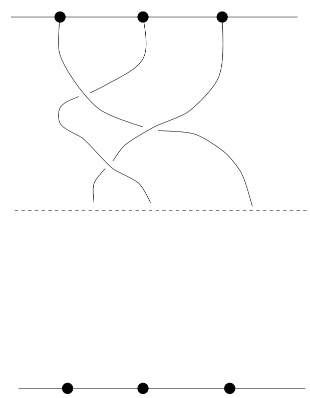
PUTNAM TRAINING PROBLEMS 2000.3
Topological Tangles and Other Knots

1. Three strings are tied to three pegs (top), and tangled as shown. How should we join three other strings to the three free ends and attach the new strings (bottom) so that the resulting strings can be untangled?



2. Prove or disprove: a closed rectangle in $\mathbb{R}^2$ can be partitioned into non-empty sets that are each homeomorphic to open intervals.

3. Prove that the unit interval $[0, 1]$ is not homeomorphic to a Cartesian product $X \times Y \subset \mathbb{R}^2$, where X and Y have at least two points.

4. (Putnam 1977.) Let C be a continuous closed curve in the plane which does not cross itself, and let Q be a point inside C. Prove that there exist points P_1 and P_2 on C such that Q is the midpoint of the line segment P_1P_2.

5. (Putnam 1975.) Does there exist a subset B of the unit circle $x^2 + y^2 = 1$ such that
   (a) B is topologically closed, and
   (b) B contains exactly one point from each pair of diametrically opposite points on the circle?

6. (Putnam 1964.) Show that the unit disk $x^2+y^2 \leq 1$ cannot be partitioned into two congruent subsets.

7. (Putnam 1964.) Into how many regions do n great circles (no three concurrent) decompose the surface of the sphere on which they lie?

8. Let P_1, P_2, ... be a sequence of distinct points which is dense in the interval $(0, 1)$. The points $P_1, \ldots P_{n-1}$ decompose the interval into n parts, and P_n decomposes one of these into two parts. Let a_n and b_n be the lengths of these two intervals. What are the possible values of the infinite sum
$$\sum_{n=1}^{\infty} a_n b_n (a_n + b_n)?$$