PUTNAM TRAINING PROBLEMS 2000.2
Fiddling with Calculus and Other Oddities

1. Let f be a real-valued function such that

   (a) f is increasing on $[0,1]$,

   (b) $f(0)=0$, and

   (c) f' exists and is increasing on $(0,1)$.

   Prove that $g(x)=f(x)/x$ is increasing on $(0,1)$.

2. (Putnam 1960.) Evaluate

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} 2^{-3k-j-(k+j)^2}$$

3. (Putnam 1999.) Let $p(x)$ be a polynomial that is non-negative for all x. Prove that, for some k, there are polynomials $f_1(x),\ldots,f_k(x)$ such that

$$p(x)=\sum_{j=1}^{k}(f_j(x))^2.$$

4. (Putnam 1968.) Prove that

$$\frac{22}{7}-\pi=\int_0^1 \frac{x^4\,(1-x)^4}{1+x^2}\,dx$$

5. (Putnam 1968.) Show that if f is real-valued and continuous on $(-\infty,+\infty)$ and $\int_{-\infty}^{+\infty} f(x)dx$ exists, then

$$\int_{-\infty}^{+\infty} f\left(x-\frac{1}{x}\right)\,dx=\int_{-\infty}^{+\infty} f(x)\,dx$$

6. (Putnam 1999.) Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)|\le C\int_{-1}^{1}|p(x)|\,dx.$$

7. (Putnam 1958.) Prove that if a unit square is partitioned into two sets, then the diameter of one of the sets is not less than $\sqrt{5}/2$. Show that no larger number will do.

   (The diameter of a set is the least upper bound of the distances between all pairs of points in the set.)