Continuous Quantum Walk

state: D — psd, trace $=1$

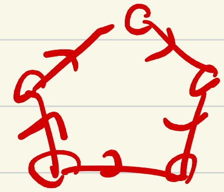
transition matrix: $U(t)$ — unitary

state at time t: $U(t) D U(-t)$

e.g graph X, adjacency matrix A

$$U(t) := \exp(itA)$$

e.g oriented graph X



skew symmetric adjacency matrix S

$$S_{i,j} = \begin{cases} 1, & i \to j \\ -1, & i \leftarrow j \end{cases}$$

$$S^T = -S$$

$$U(t) := \underbrace{\exp(tS)}_{\text{orthogonal}}$$

Uniform mixing from D

We have uniform mixing at time t, starting from D if diagonal of $D(t)$ is constant.

Usual case: $D = e_a e_a^T$ D_a vertex state

If $D_a(t) \circ I = cI$, we have local uniform mixing at a.

e.g stars $K_{1,m}$

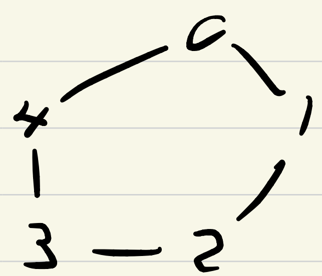
Measurements

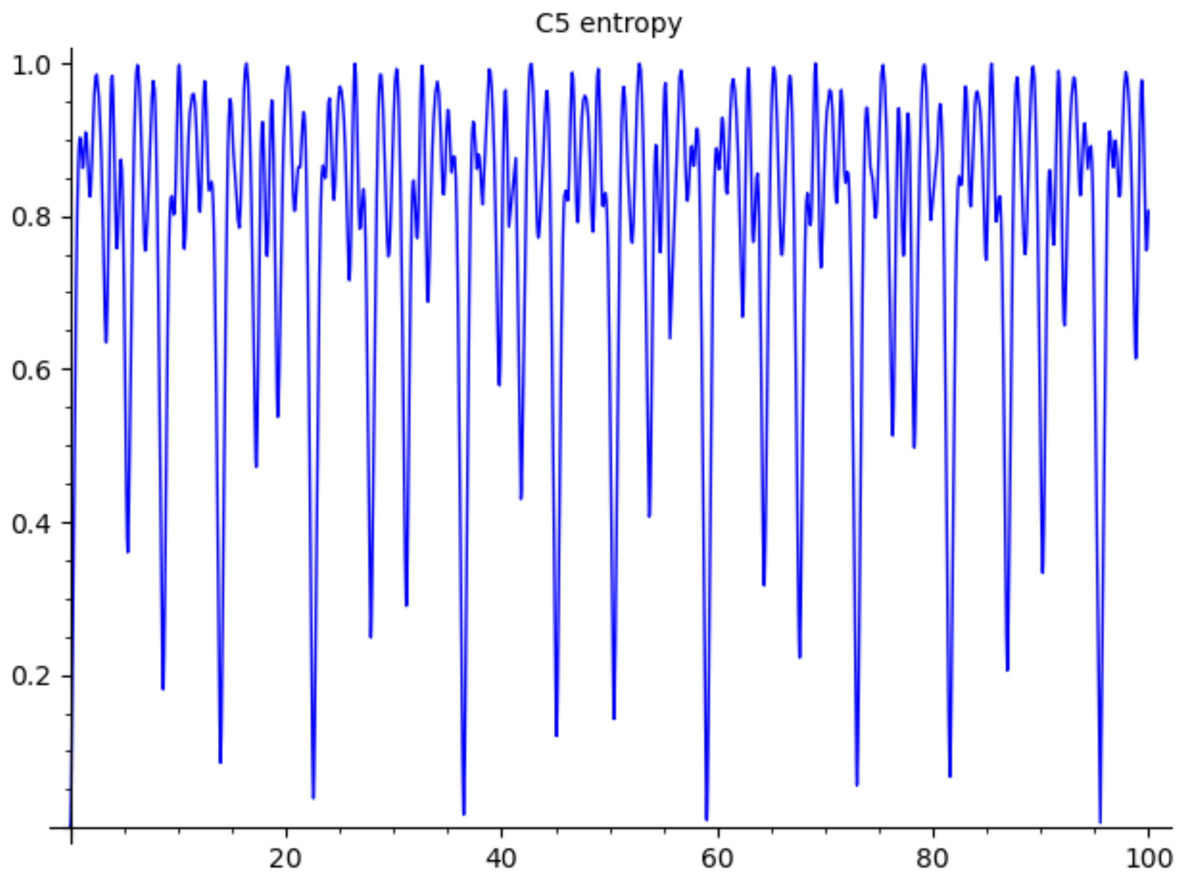
If the state of a d-dimensional quantum system is D, and we measure in the usual way, then we observe outcome i from $\{1, \ldots, d\}$ with probability $D_{i,i}$

So $D \circ I$ is a prob. density; its entropy is maximal if diagonal is constant.

Let's look for graphs with local uniform mixing:

e.g. C_5




C5 entropy
1.0
0.8
0.6
0.4
0.2
20
40
60
80
100

Algebraic States

We work with oriented graphs.

X bipartite $\quad A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$

$$\begin{pmatrix} iI & 0 \\ 0 & I \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -B \\ B^T & 0 \end{pmatrix}}_{S} \begin{pmatrix} -iI & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & -iB \\ B^T & 0 \end{pmatrix} \begin{pmatrix} -iI & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -iB \\ -iB^T & 0 \end{pmatrix} \text{ — } -iA$$

So iS & A are similar.

Compute $\exp(tS)$ using spectral decomp.

- $S^T = -S \Rightarrow (iS)^* = iS$

$$iS = \sum_r \lambda_r F_r$$

real

Hermitian

- eigenvalues of iS are symmetric about zero
idempotent for $-\lambda_r$ is $\bar{F}_r$.

- eigenvalues & entries of idempotents are algebraic numbers

$$U(t) = \exp(tS) = \sum_r e^{it\lambda_r} F_r$$

$$D_a(t) = \sum_{r,s} e^{it(\lambda_r - \lambda_s)} F_r D_a F_s$$

$$\left(U(t) e_a e_a^T U(-t)\right)_{bb} = |e_b^T U(t) e_a|^2$$

diagonal $(D_a(t))$ is flat $\Leftrightarrow$ a-column of $U(t)$ is flat

i.e. $U(t)_{b,a} = \pm \frac{1}{\sqrt{d}}$

Theorem If $D_a(t) \circ I = \frac{1}{d} I$, then the entries of D_a are algebraic numbers

$$D_a(t) = \sum_{r,s} e^{it(\lambda_r - \lambda_s)} F_r D_a F_s$$

$$F_k D_a(t) F_\ell = e^{it(\lambda_k - \lambda_\ell)} F_k D_a F_\ell$$

$\Rightarrow e^{it(\lambda_k - \lambda_\ell)}$ is algebraic

$\left(e^{it(\lambda_r - \lambda_s)}\right)^{\frac{\lambda_k - \lambda_\ell}{\lambda_r - \lambda_s}}$ is algebraic

Gelfond-Schneider

If α, β are algebraic and α^β is algebraic then either:

$$\alpha = 0, \quad \alpha = 1, \quad \beta \in \mathbb{Q}$$

So $\dfrac{\lambda_k - \lambda_\ell}{\lambda_r - \lambda_s} \in \mathbb{Q}$ RATIO CONDITION

$$\ldots\ldots \Rightarrow \frac{\lambda_k}{\lambda_r} \in \mathbb{Q} \Rightarrow \lambda_k = \underbrace{b_k}_{\text{integer}} \underbrace{\sqrt{\Delta}}_{\text{square-free integer}}$$

(Jennifer Lin)

Corollary For each integer k, there are only finitely many connected bipartite graphs that admit local uniform mixing.

with max degree k

Corollary If p is prime, $p \geq 5$, then C_p does not admit uniform mixing.

e.g. C_5, eigenvalues: $2, \frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}+1}{2}$

Examples

1) Some Cayley graphs for $\mathbb{Z}_2^d$ (Natalie Mullin)

2) " " " " $\mathbb{Z}_3^d$

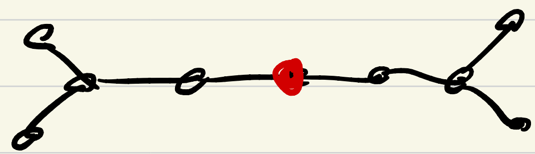
3) " " " " $\mathbb{Z}_4^d$ (Hanmeng Zhan)

4) Cartesian powers of $K_{1,3}$

5) p prime, $p \geq 5$ – cycles C_p admit ε-uniform mixing

[From now on, all joint with Xiaohong Zhang]

If there is local uniform mixing at a in the bipartite graph X, and $D_a := e_a e_a^T$, then $D_a(t)$ is a periodic function of t.



Oriented Graphs

idea: look at Cayley digraph for abelian groups

G abelian, permutations P_g ($\Rightarrow P_g P_h = P_{gh}$)

$C \subseteq G$

$A = \sum_{g \in C} P_g$ Cayley digraph

to get a graph: $e \notin C$; if $g \in C$ then $g^{-1} \in C$

$C = C^{-1}$

to get an oriented graph: $e \notin \mathcal{C}$, $\mathcal{C} \cap \mathcal{C}^{-1} = \emptyset$.

$$S = \sum_{g \in \mathcal{C}} (P_g - P_{g^{-1}})$$

$g \in \mathcal{C} \Rightarrow |g| \geq 3$

$S^T = -S$

$U(t) = \exp(tS)$ is the transition matrix for a quantum walk. It is orthogonal, not just unitary.

If we have uniform mixing at time t then

- $\sqrt{d}\, U(t)$ is a Hadamard matrix; since $U(t)$ is real, either $d=2$ or $4 \mid d$ and, since it's regular, d is a square.

- eigenvalues of S are integer multiples of $i\sqrt{\Delta}$, for some square-free integer $\Delta \geq 1$.

Question: Can we characterise the connection sets $\mathcal{C}$ for which this eigenvalue condition holds?

Bose-Mesner algebras

A **Schur polynomial** in a matrix A is a linear combination of Schur powers $A^{\circ k}$ of A.

Lemma A Schur-closed vector space of matrices has a unique basis of 01-matrices.

A **Bose-Mesner algebra** is a commutative Schur-closed matrix algebra that

(a) contains I & J

(b) closed under transpose & complex conj.

Example Abelian group G:

$$\operatorname{span}\{P_g : g \in G\}$$

is a Bose-Mesner algebra, dimension $|G|$.

Each matrix in a Bose-Mesner algebra is normal, so we can simultaneously diagonalize them.

Splitting field $\mathbb{E}$ — generated by the eigenvalues of the Schur basis.

Association scheme Schur basis of a BM-algebra.

Theorem A Bose-Mesner algebra of dimension $m+1$ has a basis of orthogonal projections $E_0, \dots, E_m$ such that:

(a) $E_0 = \frac{1}{v} J$

(b) $E_r = E_r^2 = E_r^T$

(c) $E_r E_s = 0 \quad (r \neq s)$

(d) $\sum_r E_r = I$

(e) $\exists\, p_i(j)$ such that $A_i E_j = p_i(j) E_j \quad \forall i, j$

(f) $E_r \circ E_s \in \operatorname{span}\{E_0, \dots, E_m\}$

Lemma Let $\mathcal{B}$ be a Bose-Mesner algebra with splitting field $\mathbb{E}$. If $\mathbb{F} \leq \mathbb{E}$, then the subalgebra of $\mathcal{B}$ consisting of the matrices with eigenvalues and entries in $\mathbb{F}$ is the Bose-Mesner algebra of an association scheme.

Problem: What is the Schur basis for this scheme?

e.g. $\mathbb{F} = \mathbb{Q}$

Define an equivalence relation on elements of G

$$g \sim h \iff \langle g \rangle = \langle h \rangle$$

Rational spectra...

Theorem (Bridges & Mena, 1982). Let G be abelian. A subset C of G is the connection set for a Cayley graph for G with integer eigenvalues iff it is a union of $\sim$-classes.

Bose-Mesner automorphisms A linear map $M \mapsto M^{\psi}$ is a **Bose-Mesner automorphism** if

$$(a)\ (MN)^{\psi} = M^{\psi}N^{\psi}$$

$$(b)\ (M \circ N)^{\psi} = M^{\psi} \circ N^{\psi}$$

e.g. transpose. A Bose-Mesner automorphism is necessarily invertible. arxiv:1011.1044v1

We also find that

$$(M^*)^{\psi} = (M^{\psi})^*$$

fixed → subscheme

Let $\mathcal{A}$ be an association scheme, let $\mathbb{E}$ be its splitting field and let $\mathbb{B} = \mathbb{E}[\mathcal{A}]$.

If $\sigma \in Gal(\mathbb{E}/\mathbb{Q})$, then σ permutes the projections E_s. It is not a BM-automorphism — it is not $\mathbb{E}$-linear, nonetheless if

$$M = \sum_r a_r E_r \,, \quad M^{\hat{\sigma}} := \sum a_r E_r^{\sigma}$$

then $\hat{\sigma}$ is a BM-automorphism.

Let $G \cong \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$, where $n_i | n_{i+1}$.

Assume $4 | n_k$. Define an equivalence relation $\approx_1$ on G by setting $g \approx_1 h$ if there is $\ell \in \mathbb{Z}_n^*$ with $\ell \equiv 1 \pmod 4$ and $g = h^\ell$.

Note: $g \not\approx_1 g^{-1}$, so each $\approx$-class splits in two.

We get oriented Cayley graph with eigenvalues integer multiples of i. All such Cayley graphs arise in this way.

There is a similar characterization for graphs eigenvalues with eigenvalue integer $\times \sqrt{-\Delta}$.

square-free integer ≥ 1

Problems

1) Does C_9 admit uniform mixing? C_{15}?

2) Conjecture (N. Mullin): Cayley graphs for $\mathbb{Z}_n^d$, $n \geq 5$, do not admit uniform mixing.

3) Conjecture (N. Mullin): If X admits uniform mixing at t, then e^{it} is a root of unity.

4) Is there a tree on more than four vertices that admits local uniform mixing at more than one vertex?

The End(s)

Sebi: no zebras, no elephants