CONTINUOUS-TIME ← "QUANTUM WALK + MIXING" (ALGORITHMIC LENS)

OPEN PROBLEMS IN ALGEBRAIC COMBINATORICS

X GRAPH, $L(X) =$ Laplacian matrix associated to X

ex: K_2

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = I - \sigma_x = 0 \times \frac{J}{2} + 2 \times \left(I - \frac{J}{2}\right)$$

RANDOM WALK:

$$e^{-tL(X)} = \frac{J}{2} + e^{-2t}\left(I - \frac{J}{2}\right)$$

$$\xrightarrow[t\to\infty]{\text{converges}} \frac{J}{2} = \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$$

uniform mixing

QUANTUM WALK: [FARHI-GUTMANN]

$$e^{-itL(X)} = \frac{J}{2} + e^{-2it}\left(I - \frac{J}{2}\right) = e^{-it}\begin{pmatrix} \cos t & -i\sin t \\ -i \sin t & \cos t \end{pmatrix}$$

$\boxed{t = \pi/4}$

$$\begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

exact

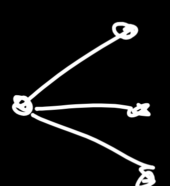
$$\text{Prob} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

X regular:

$$e^{-itL(X)} \cong e^{-itA(X)}$$

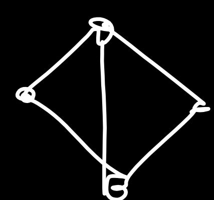
QUESTION: NON-REGULAR X (FAMILY) ∞ WITH LAPLACIAN UNIFORM MIXING

$\Rightarrow$



[ZHAN]

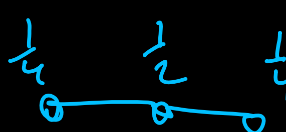
? $\cong$



$K_4 \setminus e$

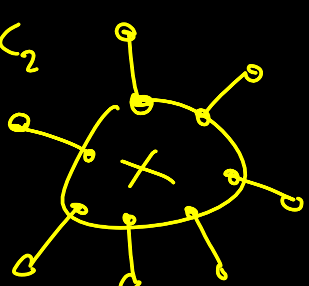
ORPHAN

$\sqrt{2}P_3$

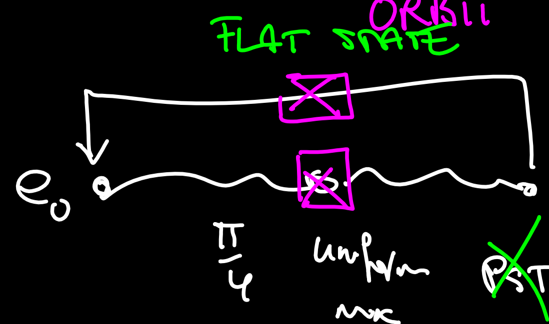


QUESTION: (UNARY) GRAPH OPERATORS THAT PRESERVE UNIFORM MIXING ?

K_2



QUESTION:



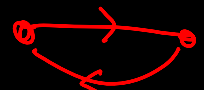
[AHARONOV, TA-SHMA] STATE GENERATION

$X = K_2$

$$\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{|0\rangle} \xrightarrow{e^{-itL(X)}} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}}_{\frac{|0\rangle - i|1\rangle}{\sqrt{2}}} \qquad \checkmark \qquad \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}_{\text{target}} ??$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

$\vec{K_2}$ i



$-i$

$$\begin{pmatrix}1\\0\end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$e^{it\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}$$

$$\begin{pmatrix}\cos^2 t\\ \sin^2 t\end{pmatrix} \leftarrow \begin{pmatrix}\cos t\\ -i\sin t\end{pmatrix}$$

PROB DISTR $P \in [0,1]^{V(X)}$

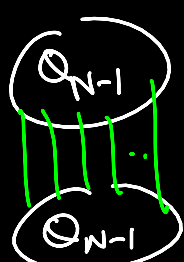
$$|\psi\rangle = \sum_{a\in V(X)} \sqrt{P_a}\,|a\rangle \qquad \sum_{a\in V(X)} P_a = 1$$

GENERALIZE K_2 : 3 WAYS

① $\underline{Q_N}$ ② $\underline{K_N}$ P_N C_N ③ $\underline{K_{1,N}}$

[MOORE, RUSSELL]

① N-CUBES



$$\underline{Q_N} = \begin{pmatrix} A(Q_{N-1}) & I \\ I & A(Q_{N-1})\end{pmatrix} \rightarrow \sum \begin{matrix} \sigma_Y \otimes I \otimes \cdots \otimes I \\ I \otimes \sigma_Y \otimes \cdots \otimes I \\ \vdots \\ I \otimes I \otimes \cdots \otimes \sigma_Y \end{matrix}$$

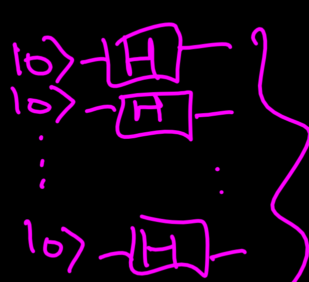
oriented N-cube

commuting

$$e^{-itA(Q_N)} = \Big(\underbrace{e^{itA(K_2)}}\Big)^{\otimes n} \qquad t = \pi/4$$

$$\underline{\begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2}\end{pmatrix}^{\otimes n}} \quad \times$$

$$\underline{|\psi\rangle} = \frac{1}{\sqrt{2^N}} \sum_{k=0}^{2^N-1} |k\rangle$$



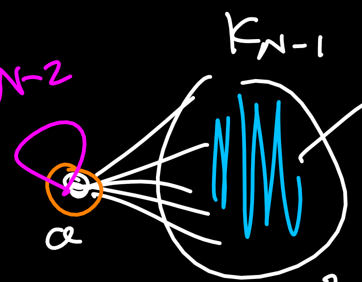
MIXING SPEEDUP

RANDOM WALK: $P = \underbrace{\frac{1}{2} I}_{\text{lazy}} + \cancel{\frac{1}{2}} \underbrace{\frac{1}{N} A(Q_N)}_{\text{stochastic}} \longrightarrow \underline{\Omega(N \log_2 N)}$

QUANTUM WALK: $e^{-it\frac{A(Q_N)}{N}}$, $N\frac{\pi}{4} = \underline{O(N)}$

QUESTION:
FASTER SPEEDUP?
EXPONENTIAL ~ ?

2) CLIQUES: K_N



$e^{-itA(K_N)} \approx I$

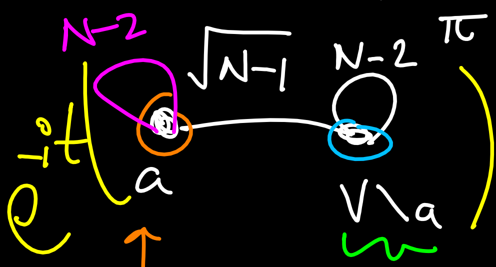
$\frac{1}{\sqrt{N-1}}\begin{pmatrix}0\\1\\ \vdots\\1\end{pmatrix} \approx \frac{1}{\sqrt{N}}\begin{pmatrix}1\\ \vdots\\1\end{pmatrix}$

QUOTIENT

2-cell Equitable partition

[GODSIL] SEDENTARY GRAPHS

QUESTION: ARE ALL EXPANDERS SEDENTARY?



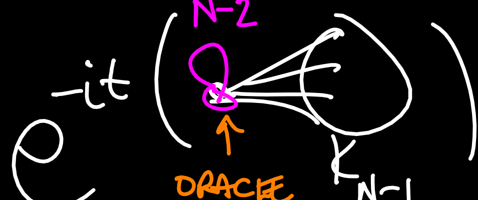
$e^{-it(\ldots)} \cong e^{-it(\ldots)}$

$t = \sqrt{N-1}\,\frac{\pi}{2}$

How DO WE ESCAPE a?

PERFECT STATE TRANSFER $t = \pi/2$

[FARHI-GUTMANN]



marked vertex

$\begin{pmatrix}1\\0\end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{N}}\begin{pmatrix}1\\ \vdots\\1\end{pmatrix}$

GROVER SEARCH

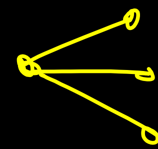
REVERSE MIXING

3) CLAWS

MIXING GAME

ALICE   GRAPH X BOB

K_2: 

$K_{1,3}$ 

1) CHOOSE $p \in (0,1]^{V(X)}$ (FULL SUPPORT)

2) CHOOSE WEIGHT $w \in \mathbb{R}_+^{E(X)}$

STATE GENERATION

BOB WINS IFF $\exists t$

$$\left|\langle e_b, e^{-itA(X_w)} e_a \rangle\right| = \sqrt{p_b} \quad \forall b \in V(X)$$

$\boxed{\forall a}$ vs $\boxed{\exists a}$

"K_2"

CLAIM [CARLSON ET AL] BOB WINS ON $K_{1,N}$

"TREES"

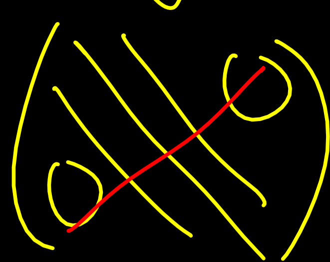
QUESTION: WHAT OTHER GRAPHS WHERE BOB WINS?

P_N? → [KAY]

↓

$t = \pi$

⇓



→ $\boxed{0}$

→ $2\mathbb{Z}+1$

$\lambda_1 > \lambda_2 > \dots > \lambda_n$

⇓

Jacobi matrix unique