Quantum Walks on Finite Groups

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Joint work with Julien Sorci.

Overview

Background. Cayley Graphs, Characters

Strong Cospectrality

Perfect State Transfer

Examples

Uniform mixing

Open Problems

Continuous-time quantum walk

Let A be the adjacency matrix of a graph Γ. Then a continuous-time quantum walk on Γ is defined by the family of unitary operators

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acting on $\mathbb{C}V(\Gamma)$.

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acting on $\mathbb{C}V(\Gamma)$.

Γ has **perfect state transfer** from a to $b \in V(\Gamma)$ at time τ if $|U(\tau)_{b,a}| = 1$.

Γ has **instantaneous uniform mixing** at time τ if for all a, $b \in V(\Gamma)$ we have $|U(\tau)_{a,b}| = \frac{1}{\sqrt{|V(\Gamma)|}}$.

Basic questions: Which graphs admit PST and IUM? Examples? Nec./suff conditions?

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Eigenvalues come from **Irreducible characters**. $\chi \in \mathrm{Irr}(G)$ gives the eigenvalue

$$\theta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s), \qquad \text{with } \theta_1 = |S|.$$

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Idempotents of scheme. View g either as an element of $\mathbb{C}G$ or as a $|G| \times |G|$ matrix under the regular representation.

$$E_\chi = \frac{\chi(1)}{|G|} \sum_g \chi(g^{-1})g$$

For each eigenvalue θ, let $X(\theta) = \{\chi \in \mathrm{Irr}(G) \mid \theta_\chi = \theta\}$. Then $\tilde{E}_\theta = \sum_{\chi \in X(\theta)} E_\chi$ is the idempotent of θ.

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Proof.

Suppose $\tilde{E}_\theta h = \sigma_\theta \tilde{E}_\theta g$, $\sigma_\theta \in \{1, -1\}$. Let f be a polynomial with $f(\theta) = \sigma_\theta$ for all eigenvalues θ. Then from

$$A = \sum_\theta \theta \tilde{E}_\theta$$

we get

$$f(A) = \sum_\theta \sigma_\theta \tilde{E}_\theta,$$

and so $f(A)^2 = I$ and $f(A)g = h$. Then $f(A) = hg^{-1} \in Z(\mathbb{C}G) \cap G$ must be a central involution. □

Strong Cospectrality in terms of characters.

Theorem

Distinct elements g and h of G are strongly cospectral iff there is a central involution z such that the following hold.

(a) $h = zg$.

(b) $(\forall \theta)$, $(\forall \chi, \psi \in X(\theta))$, $\chi(z)/\chi(1) = \psi(z)/\psi(1)$.

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Theorem

In $\mathrm{Cay}(G, S)$ we have PST between vertices g and h at some time if and only if the following hold.

(a) *The eigenvalues are integers.*

(b) *g and h are strongly cospectral.*

(c) *Let $z = hg^{-1}$ and let $\Phi^+ = \{\theta_\chi | \chi(z) > 0\}$ and $\Phi^- = \{\theta_\chi | \chi(z) < 0\}$. There is an integer N such that*

   (i) *for all $\theta_\chi \in \Phi^-$, $v_2(\theta_1 - \theta_\chi) = N$; and*

   (ii) *for all $\theta_\chi \in \Phi^+$, $v_2(\theta_1 - \theta_\chi) > N$.*

Minimum time

Minimum value of t for PST is $2\pi/g$, where $g = \mathrm{gcd}\{\theta_1 - \theta_\chi \mid \chi \in \mathrm{Irr}(G)\}$.

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Lemma

Any common divisor of the $\theta_1 - \theta_\chi$ divides $|G|$ (as algebraic integers).

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Lemma

Any common divisor of the $\theta_1 - \theta_\chi$ divides $|G|$ (as algebraic integers).

- No assumption of integrality. Proof is similar to abelian case (Cao-Feng-Tan).

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Extraspecial Groups

Let p be a prime. A p-group G is extraspecial if $Z(G)$ has order p and $G/Z(G)$ is elementary abelian. Structure is known, G is a central product of extraspecial groups of order p^3, and for each p there are just two isomorphism types. When $p = 2$, we have D_8 and Q_8.

Characters

Let G be extraspecial of order 2^{2n+1}, with $Z(G) = \langle z \rangle$.

Irreducible characters of a central product are products of irreducible characters of the component groups such that the factors in the product all agree on the amalgamated central subgroup.

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- Irreducible characters of a central product are products of irreducible characters of the component groups such that the factors in the product all agree on the amalgamated central subgroup.
- So G has a unique nonlinear character Ψ, and we have $\Psi(1) = 2^n$, $\Psi(z) = -2^n$, $\Psi(g) = 0$ if $g \notin Z(G)$.

Character Table of D_8/Q_8

$X.1$	1	1	1	1	1
$X.2$	1	1	-1	1	-1
$X.3$	1	1	1	-1	-1
$X.4$	1	1	-1	-1	1
$X.5$	2	-2	0	0	0

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$G/Z(G) \cong \mathbb{F}_2^{2n}$ and each $y \in \mathbb{F}_2^{2n}$ gives a character $\lambda_y(x) = (-1)^{x \cdot y}$. Let $\overline{S}$ be the image of S in $G/Z(G)$. Let $e_y = \#\{x \in \overline{S} \mid x \cdot y = 0\}$.

$\theta_{\lambda_y} = 2 \sum_{x \in \overline{S}} (-1)^{x \cdot y} = 2(e_y - (\ell - e_y)) = 4e_y - 2\ell$.

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The precise conditions on S for PST can been worked out.

Heisenberg Groups

Let $G = H_n(\mathbb{F}_q)$ be the group of matrices of the form

$$\begin{bmatrix} 1 & v^t & a \\ 0 & I_n & w \\ 0 & 0 & 1 \end{bmatrix}, \quad v, w \in \mathbb{F}_q^n, a \in \mathbb{F}_q.$$

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Noncentral conj. classes have size q and are the cosets $gZ(G)$

Characters

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- For each nonprincipal character μ of $Z(G)$ there is a character Ψ_μ whose restriction to $Z(G)$ is $q^n\mu$ and which vanishes on $G \setminus Z(G)$.

Character table of H_1(4)

2	6	4	4	4	6	6	6	4	4	4	4	4	4	4	4	4	4	4	4
	1a	2a	2b	2c	2d	2e	2f	2g	4a	4b	4c	2h	4d	4e	4f	2i	4g	4h	4i
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X.5	1	1	-1	-1	1	1	1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
X.6	1	1	-1	-1	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
X.7	1	1	-1	-1	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
X.8	1	1	-1	-1	1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
X.9	1	-1	1	-1	1	1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
X.10	1	-1	1	-1	1	1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
X.11	1	-1	1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
X.12	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
X.13	1	-1	-1	1	1	1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
X.14	1	-1	-1	1	1	1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
X.15	1	-1	-1	1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
X.16	1	-1	-1	1	1	1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
X.17	4	.	.	.	4	-4	-4	.	.	.	.	.	.	.	.	.	.	.	.
X.18	4	.	.	.	-4	-4	4	.	.	.	.	.	.	.	.	.	.	.	.
X.19	4	.	.	.	-4	4	-4	.	.	.	.	.	.	.	.	.	.	.	.

Check for PST

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$$\theta_1 - \theta_\chi \equiv \begin{cases} 0 \pmod q & if \theta_\chi \in \Phi^+ \\ 2 \pmod q & if \theta_\chi \in \Phi^- \end{cases}$$

Hence condition for PST is satisfied.

Suzuki 2-groups

Let $n = 2m + 1$ be odd and let $F \in \mathrm{Aut}(\mathbb{F}_{2^n})$ be the Frobenius map $F(x) = x^2$ Then $\sigma = F^{m+1}$ satisfies $\sigma^2 = F$. Let $G = S(2^n)$ be the group of matrices

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & \sigma(x) \\ 0 & 0 & 1 \end{bmatrix}, \quad x \in \mathbb{F}_{2^n}.$$

$|Z(G)| = |G/Z(G)| = 2^n$, all involutions lie in $Z(G)$.
Similar analysis to Heisenberg case shows that PST holds for many sets S. (Exercise)

Character table of S(8)

	1a	2a	2b	2c	2d	2e	2f	2g	4a	4b	4c	4d	4e	4f	4g	4h	4i	4j	4k	4l	4m	4n
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X.5	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1
X.6	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1
X.7	1	1	1	1	1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1
X.8	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1
X.9	2	2	-2	-2	-2	2	2	-2	.	.	.	.	A	-A	.	.	.	.	.	.	.	.
X.10	2	2	-2	-2	-2	2	2	-2	.	.	.	.	-A	A	.	.	.	.	.	.	.	.
X.11	2	-2	2	2	-2	2	-2	-2	A	-A	.	.	.	.	.	.	.	.	.	.	.	.
X.12	2	-2	2	2	-2	2	-2	-2	-A	A	.	.	.	.	.	.	.	.	.	.	.	.
X.13	2	-2	-2	-2	2	2	-2	2	.	.	.	.	.	.	.	.	.	.	.	.	-A	A
X.14	2	-2	-2	-2	2	2	-2	2	.	.	.	.	.	.	.	.	.	.	.	.	A	-A
X.15	2	2	2	-2	2	-2	-2	-2	.	.	.	.	.	.	.	.	-A	A	.	.	.	.
X.16	2	2	2	-2	2	-2	-2	-2	.	.	.	.	.	.	.	.	A	-A	.	.	.	.
X.17	2	2	-2	2	-2	-2	-2	2	.	.	-A	A	.	.	.	.	.	.	.	.	.	.
X.18	2	2	-2	2	-2	-2	-2	2	.	.	A	-A	.	.	.	.	.	.	.	.	.	.
X.19	2	-2	2	-2	-2	-2	2	2	.	.	.	.	.	.	.	.	.	.	A	-A	.	.
X.20	2	-2	2	-2	-2	-2	2	2	.	.	.	.	.	.	.	.	.	.	-A	A	.	.
X.21	2	-2	-2	2	2	-2	2	-2	.	.	.	.	.	.	A	-A	.	.	.	.	.	.
X.22	2	-2	-2	2	2	-2	2	-2	.	.	.	.	.	.	-A	A	.	.	.	.	.	.

A = 2*E(4) = 2*Sqrt(-1) = 2i

Overview

Background. Cayley Graphs, Characters

Strong Cospectrality

Perfect State Transfer

Examples

Uniform mixing

Open Problems

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$$(\forall g \in G) \quad |\sum_\chi e^{i\tau\theta_\chi} \chi(1)\chi(g)| = \sqrt{|G|}. \tag{1}$$

The above is a condition on the columns of the character table. There is a "dual" condition on the rows (Chan): IUM occurs at time τ iff

$$(\exists t_i \in \mathbb{C}, |t_i| = 1, t_{i^*} = t_i) \quad (\forall \chi) \quad \sqrt{|G|} e^{i\tau\theta_\chi} = \sum_i t_i \frac{\chi(K_i)}{\chi(1)}. \tag{2}$$

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Conditions (1) and (2) are related: If the t_i exist then,

$$\sqrt{|G|} t_i = \sum_\chi e^{i\tau\theta_\chi} \chi(1)\chi(g_i)$$

Complex Hadamard matrices

Similarly, $Z(\mathbb{C}G)$ contains a complex Hadamard matrix iff one of the follwing dual conditions holds.

$$(\exists t_i \in \mathbb{C}, |t_i| = 1)(\forall \chi) \quad \sqrt{|G|} = |\sum_i t_i \frac{\chi(K_i)}{\chi(1)}|. \tag{3}$$

$$(\exists u_\chi \in \mathbb{C}, |u_\chi| = 1)(\forall g) \quad \sqrt{|G|} = |\sum_\chi u_\chi \chi(1)\chi(g)|. \tag{4}$$

Apply to examples

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Condition (3) immediately implies $|\mathrm{Supp}(\chi)| \geq \sqrt{|G|}$.
Let G be an extraspecial p-group or a finite Heisenberg group.
Then G has a character supported on $Z(G)$ and
$|Z(G)| < \sqrt{|G|}$, so there is no complex Hadamard matrix in
$Z(\mathbb{C}G)$, hence no IUM at any time for any $\mathrm{Cay}(G, S)$.

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Suzuki 2-groups cannot be eliminated this way; there is a complex Hadamard matrix in $Z(\mathbb{C}G)$.
But no IUM at any time t. This is because in condition (1) χ and $\overline{\chi}$ give same eigenvalue.

What examples have been found?

Examples of IUM on Cayley graphs:
cubelike graphs, halved and folded cubes (Chan)
cubelike graphs from bent functions, integral abelian Cayley graphs (Cao-Feng-Tan).
No nonabelian examples known.

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- Complex Hadamard matrices in $Z(\mathbb{C}G)$ for nonabelian G.
- More PST examples in nonabelian groups (known in 2-groups, dihedral, direct products)