Some Open Questions in Erdős-Ko-Rado Combinatorics

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Abstract

Some open questions related to Erdős-Ko-Rado combinatorics for the "Open Problems in Algebraic Combinatorics Workshop".

1-Skeletons

For any graph $G = (V, E)$, let $PM(G)$ denote the perfect matching polytope of G. The *1-skeleton* of a polytope is graph given by the vertices and edges of the polytope. It is well-known that two perfect matchings m, m' of G are joined by an edge in $PM(G)$ if and only if their symmetric difference $m\Delta m'$ is a single cycle. If $G = K_{2n}$, then the 1-skeleton Γ of $PM(K_{2n})$ is a union of associates of *the perfect matching association scheme*

$$\Gamma = \sum_{k=0}^{n-2} A_{(n-k,1^k)}.$$

Open Question 1: What is the independence number of Γ?

This question is inspired by Kane et al.'s work on the independence number of the 1-skeleton of $PM(K_{n,n})$. Graphs coming from polytopes tend to have nice expansion and pseudorandomness properties, which doesn't work well with the Delsarte–Hoffman ratio bound. Structure vs. randomness techniques are better suited for this task, giving significantly better upper bounds. Constructing large independent sets in Γ is also a difficult question.

The associate $A_{(2,1^{n-2})}$ is sometimes called the perfect matching "flip graph". Determining the chromatic number $\chi(A_{(2,1^{n-2})})$ of this graph has received some attention (see, for example, Fabila-Monroy et al. and Cioaba et al.). Fabila-Monroy et al. also pose the question of determining $\chi(\Gamma)$. A good upper bound on the independence number of Γ would make progress on this question.

Open Question 2: What is the clique number of Γ?

This doesn't seem any easier, but finding large cliques would make progress on Open Question 1 via the clique-coclique bound.

Erdős-Ko-Rado for Tabloids

Let $\lambda = (\lambda_1, \cdots, \lambda_\ell) \vdash n$ be an integer partition of n. A λ-*tabloid* is an ordered partition of $\{1, 2, \cdots, n\}$ into ℓ sets such that the first set has size λ_1, the second set has size λ_2, and so on. For example, we can represent k-sets, partial permutations (injections), and full permutations as $(n-k, k)$-tabloids, $(n-k, 1^k)$-tabloids, and (1^n)-tabloids respectively.

Open Question 3: Give an Erdős-Ko-Rado Theorem for λ-tabloids for all $\lambda \vdash n$, and show that the largest intersecting families are the canonically intersecting families.

If one can give an algebraic proof of this via the ratio bound, then there should be a cute proof of the characterization of the extremal families via the b-matching polytope of $K_{n,n}$ where the b-vector depends on λ (see Godsil and Meagher's textbook for more details).

t-Intersecting Families and q-Analogues

Open Question 4: Give t-intersecting Erdős-Ko-Rado results for q-analogues of domains for which t-intersecting Erdős-Ko-Rado results hold for sufficiently large n (e.g., permutations, perfect matchings).

This question was the main focus of my talk, so see my slides for more details.