Erdős-Ko-Rado Theorems for Groups

Karen Meagher
(joint work with Bahman Ahmadi, Chris Godsil, Sarobidy Razafimahatratra, Peter Sin, Pablo Spiga and Pham Huu Tiep)

Department of Mathematics and Statistics
University of Regina

Intersecting Permutations

Let G be a permutation group —so we have a group G with an action.

- Two permutations $\sigma, \pi \in G$ are *intersecting* if for some $i \in \{1, \ldots, n\}$.

$$\sigma(i) = \pi(i) \quad \text{or} \quad \pi^{-1}\sigma(i) = i.$$

- A permutation is a *derangement* if it fixes no points.
- Permutations σ and π are intersecting if and only if $\pi^{-1}\sigma$ is **not** a derangement.
- *A set S is intersecting* if any two elements from S are intersecting.

The question

What is the size of the largest set of intersecting permutations in a group?

What is the structure of the largest set of intersecting permutations in a group?

Group Action

- Consider a group G with a transitive, faithful action on a set Ω. (no two distinct permutations have the same action on all points)
- Intersection depends on the group action.
- Usually consider common group actions, such as $\mathrm{Sym}(n)$ on $\{1, 2, \ldots, n\}$, or on ordered pairs of $\{1, 2, \ldots, n\}$.

For any transitive group action of G, there is a $H \leq G$ such that the action is equivalent to the transitive action of G on G/H.
(H is the stabilizer of a point in Ω.)

Group Action

Assume the group action is G on G/H.
If $g \in G$ has a fixed point, then for some $x \in G$

$$g\,xH = xH \quad \rightarrow \quad g \in xHx^{-1}.$$

The set of derangements is

$$G - \bigcup_{x \in G} xHx^{-1}.$$

Bardestani and Mallahi-Karai noted ff S is an intersecting set then

$$SS^{-1} \subseteq \bigcup_{x \in G} xHx^{-1}$$

1. H is intersecting
2. If H is normal then any intersecting set is a subset of H (or a coset of H) so H is the only intersecting set of maximum size.

Open Question

For which H does $\bigcup_{x \in G} xHx^{-1}$ contain a subgroup, other than H?

Intersecting Sets

Proposition

If S is intersecting set in G, then σS and $S\sigma$ are also intersecting for any $\sigma \in G$.

(We can always assume that the identity is in S.)

Proposition

Any subgroup $S \leq G$ in which every element has a fixed point is intersecting.

Proof. If $\sigma, \pi \in S$, then $\sigma\pi^{-1} \in S$.
So $\sigma\pi^{-1}$ has a fixed point and σ and π are intersecting.

Canonical Intersecting Sets

The *canonical intersecting sets* are

$$S_{i,j} = \{\sigma \in \mathrm{Sym}(n) \mid \sigma(i) = j\}.$$

1. If $i = j$, then $S_{i,i}$ is the stabilizer of i (this is a subgroup),
2. if $i \neq j$ the $S_{i,j}$ is a coset of a subgroup.
3. $S_{i,j}$ is an intersecting set of size $(n-1)!$.
4. Use $v_{i,j}$ for the characteristic vector of $S_{i,j}$.

For which groups are the canonical intersecting sets the largest intersecting sets?

Three EKR Properties

EKR-property

A group action has the *EKR-property* if the **size** of the largest set of intersecting permutations is the size of the largest stabilizer of a point.

EKR-module Property

A group action has the *EKR-module property* if the characteristic vector of any intersecting set in G of maximum size is in the module

$$V = \mathrm{span}\{v_{i,j}, \mid i,j = 1, 2, \dots, n\}.$$

The EKR-module property implies that for any maximum intersecting set S the characteristic vector of v_S is a linear combination of $v_{i,j}$.

Strict-EKR Property

A group action has the *strict-EKR* property if the **only maximum** intersecting permutations are the stabilizer of a point or a coset of one.

Derangement Graph

For any $G \leq \mathrm{Sym}(n)$ we can define a *Derangement Graph*.

- Γ_G denotes the derangement graph for a group G.
- The vertices are the elements of G.
- Vertices $\sigma, \pi \in G$ are adjacent if and only if $\pi^{-1}\sigma$ is a derangement.
  (So permutations are adjacent if they are **not** intersecting.)

An intersecting set in G is a coclique in Γ_G.

General Examples of Derangement Graphs

- If $G = \langle \sigma \rangle$ where σ is an n-cycle, then Γ_G is a complete graph.
- If $G = \langle \sigma \rangle$, then Γ_G is the complement of the circulant graph $(\mathbb{Z}_{|G|}, C)$ where C is the set of all multiples of the cycle lengths of σ.

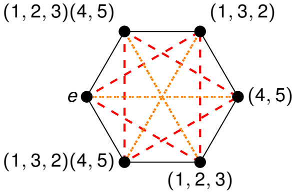

Figure: Derangement graph for Γ_G where $G = \langle (1, 2, 3)(4, 5) \rangle$.

- If $G \times H$ is the **internal** direct product, then $\Gamma_{(G \times H)} = \Gamma_G \times \Gamma_H$.

Open Question

Are other group products interesting?

Facts about the Derangement Graph

1. The derangement graph is a *normal* Cayley graph

$$\Gamma_G = \mathrm{Cay}(G, \mathrm{der}(G))$$

2. The *connection set* of the Cayley graph is set of derangements; so it is closed under conjugation.
3. Γ_G is a graph in an association scheme.

Γ_G is connected if and only if $\mathrm{der}(G)$ generates G.

Open Question

For which groups G does $\mathrm{der}(G)$ generate the group.

Eigenvalues of Cayley Graphs

Theorem

If $Cay(G, C)$ is a normal Cayley graph, then its eigenvalues are

$$\frac{1}{\chi(1)} \sum_{\sigma \in C} \chi(\sigma)$$

where χ is an irreducible character of G.

For each irreducible representation χ of G there is

1. an eigenvalue λ_χ,
2. a G-module, and
3. a projection to the module E_χ.

Open Question

Find and prove the interesting patterns in the eigenvalues for $\Gamma_{\mathrm{Sym}(n)}$ (Ku and Wales have a paper on this).

Clique-Coclique Bound

Theorem (Clique-Coclique Bound)

For a graph X in an association scheme

$$\alpha(X)\omega(X) \leq |V(X)|.$$

Assume equality holds and S is a maximum coclique and C a maximum clique.

1. *then $|S \cap C| = 1$.*
2. *for every irreducible representation χ either*

$$E_\chi v_C = 0 \quad or \quad E_\chi v_S = 0.$$

Example with Clique-Coclique

For any n the graph has a $\Gamma_{\mathrm{Sym}(n)}$ has an n clique.
(The group $C = \langle (1, 2, \ldots, n) \rangle$ is a clique)

$$\alpha(\Gamma_{\mathrm{Sym}(n)}) \leq \frac{n!}{n} = (n-1)!$$

Corollary

If a group has a sharply 1-transitive subgroup, then the group has the EKR property.

Corollary

The symmetric group has the EKR property.

Example with Clique-Coclique Bound

For every irreducible representation χ_λ of $\mathrm{Sym}(n)$ except $\lambda = [n]$ and $\lambda = [n-1, 1]$, there is a sharply transitive subgroup C so that $\chi_\lambda(C) \neq 0$.

Theorem

The symmetric group has the EKR-module property.

Open Question

Is there a group property that would imply that there are enough big cliques for the group to have the EKR module property?

Example with Clique-Coclique Bound

Theorem (Wang and Zhang)

$\mathrm{Sym}(n)$ *has strict EKR.*

Open Question

Can the Wang and Zhang proof be applied to other groups?

Theorem (Wang and Zhang)

The Coxeter groups of types S_n^B *and* S_n^D *with* $n > 3$ *have strict EKR.*

Graph Homomorphisms

Lemma

The fractional chromatic number of Γ_G *is* $\frac{|G|}{\alpha(\Gamma_G)}$

If there is a graph homomorphism (a map $V(H)$ to $V(G)$ such that if x, y are adjacent in H, their images are adjacent in G.)

$$\Gamma_H \to \Gamma_G$$

then

$$\chi_f(\Gamma_H) = \frac{|H|}{\alpha(\Gamma_H)} \leq \frac{|G|}{\alpha(\Gamma_G)} = \chi_f(\Gamma_G)$$

So we get the following bound

$$\alpha(\Gamma_G) \leq \frac{|G|}{|H|}\alpha(\Gamma_H).$$

Graph Homomorphisms

Theorem

If $H \leq G$ and H is transitive and H has the EKR property, then G has the EKR property.

Proof. By embedding

$$\Gamma_H \to \Gamma_G$$

So

$$\alpha(\Gamma_G) \leq \frac{|G|}{|H|}\alpha(\Gamma_H) =\leq \frac{|G|}{|H|}\frac{|H|}{n} = \frac{|G|}{n}.$$

Theorem

If $H \leq G$ and H is 2-transitive and H has the strict EKR property, then G has the strict EKR property.

Open Question

How can this be used other than with $H < G$?

Hoffman's Ratio Bound

Ratio Bound

If X is a d-regular graph then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where d is the degree and τ is the least eigenvalue for the adjacency matrix for X.

If

- equality holds in the ratio bound
- and y is a characteristic vector for a maximum coclique,

then

$$y - \frac{\alpha(X)}{|V(X)|}\mathbf{1}$$

is an eigenvector for τ.

Frobenius groups

Example

If $G \leq \mathrm{Sym}(n)$ is a Frobenius group, then the spectrum of Γ_G is

$$\{n-1^{(k)}, \quad -1^{k(n-1)}\}.$$

Γ_G is k copies of the complete graph on n vertices.

- There are n^k maximum coclique, so not strict EKR for $k > 2$.
- Eigenspace for -1 has dimension $k(n-1)$;
- this is the dimension of the span of $v_{i,j}$.
- (Pantagi) Every Frobenius groups has the EKR module property.

2-Transitive Subgroups

1. If $G \leq \mathrm{Sym}(n)$ is a transitive subgroup, then the stabilizer of a point has size $|G|/n$.
2. The value of the permutation character minus the trivial character is $\chi(g) = \mathrm{fix}(g) - 1$.
3. If G is 2-transitive, then the character χ is irreducible.
4. If $\mathrm{der}(G)$ is the set of derangements in G, then

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{g \in \mathrm{der}(G)} \chi(g) = \frac{-|\mathrm{der}(G)|}{n-1}$$

is an eigenvalue.

Least Eigenvalues of Derangement Graph

If $\frac{-|\operatorname{der}(G)|}{n-1}$ is the **least** eigenvalue for Γ_G then the ratio bound implies

$$\alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{|\operatorname{der}(G)|}{\frac{-|\operatorname{der}(G)|}{n-1}}} = \frac{|G|}{n}.$$

So G has the EKR property!

Open Question

For which 2-transitive groups does the character χ give the least eigenvalue of the derangement graph?

Open Question

What's up with the 1-transitive groups that have $\frac{-|\operatorname{der}(G)|}{n-1}$ as a least eigenvalue?

Example: $\mathrm{PGL}(2, q)$

Example

Let $G = \mathrm{PGL}(2, q)$, the characters can be calculated:

Character	λ_1	λ_{-1}	ψ_1	ψ_{-1}	η_β	ν_γ
Eigenvalue	$\frac{q^2(q-1)}{2}$	$\frac{-q(q-1)}{2}$	$\frac{-q(q-1)}{2}$	$\frac{q-1}{2}$	q	0

Theorem (M. Spiga)

$\mathrm{PGL}(2, q)$ *has the EKR property.*

Improved Ratio bound

Theorem

Sharpened Ratio Bound Let Γ_G *is the derangement graph G and d its degree. Let* λ_χ *be the least eigenvalue of the adjacency matrix of* Γ_G *over all irreducible representations* χ *for which* $E_\chi v_S \neq 0$ *for S a maximum coclique in* Γ_G*. Then*

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\lambda_\chi}}$$

Theorem (M. and Spiga; Spiga)

$\mathrm{PGL}(n, q)$ *for* $n \geq 2$ *has the EKR module property.*

Open Question

How else can the ratio bound be sharpened?

Weighted Adjacency Matrices

A *weighted* adjacency matrix for a graph X is a

1. $|V(X)| \times |V(X)|$
2. symmetric matrix with
3. the (i,j)-entry non-zero only it vertices i and j are adjacent in X.
4. Put a weight on the edges, can weight them with 0.

Ratio Bound

If X is a d-regular graph then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where d is the degree and τ is the least eigenvalue for a weighted adjacency matrix of X.

Weightings for Derangement Graphs

A derangement graph Γ_G is the union of Cayley graphs in which each connection set is a conjugacy class of derangements.

Lemma

If there is weighting for a derangement graph for which the ratio bound holds with equality, then there is a weighting that is constant on the conjugacy classes for which the ratio bound holds with equality

$$A(\Gamma_G) = \sum_{\substack{C \\ \text{conjugacy class}}} w_C \, A(Cay(G, C)).$$

Open Question

Does such a weighted exist for all 2-transitive groups?

$PSU(3, q)$

Partial Character Table for $PSU(3, q)$ (for $gcd(3, q+1) = 1$).

	number	size	trivial	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
						$1 \leq u \leq q+1$	$1 \leq u \leq q+1$			
			1	$q(q-1)$	q^3	q^2-q+1	$q(q^2-q+1)$	$(q-1)(q^2-q+1)$	$(q+1)(q^2-q+1)$	$(q+1)^2(q-1)$
C_1	$\frac{q^2-q}{3}$	$\frac{\lvert G\rvert}{q^2-q+1}$	1	-1	-1	0	0	0	0	B
C_2	$\frac{q^2-q}{6}$	$\frac{\lvert G\rvert}{(q+1)^2}$	1	2	-1	$e^{3uk}+e^{3ul}+e^{3um}$	$-1(e^{3uk}+e^{3ul}+e^{3um})$	$-1\sum_{[u,v,w]} e^{uk+vl+wm}$	0	0

1. There are two families of conjugacy classes of derangements.
2. Weigh these conjugacy classes so the eigenvalues for χ_1 and χ_2 are both -1.
3. The character χ_3, with $u = (q+1)/2$, also gives an eigenvalue of -1
4. $PSU(3, q)$ has the EKR-module property.

EKR Property for 2-transitive groups

Theorem (M, Spiga, Tiep)

All two transitive groups have the EKR property.

First we used the two reductions:

1. if a group has a sharply 1-transitive subgroup then it has the EKR property.
2. if G has a transitive subgroup H with the EKR property, then G has the EKR property.

We only needed to look at minimal transitive subgroups of almost simple type.

Table of 2-transtiive groups of almost simple type

Line	Group S	Degree	Condition on G	Remarks
1	$\mathrm{Alt}(n)$	n	$\mathrm{Alt}(n) \leq G \leq \mathrm{Sym}(n)$	$n \geq 5$
2	$\mathrm{PSL}_n(q)$	$\frac{q^n-1}{q-1}$	$\mathrm{PSL}_n(q) \leq G \leq \mathrm{P\Gamma L}_n(q)$	$n \geq 2$, $(n,q) \neq (2,2),(2,3)$
3	$\mathrm{Sp}_{2n}(2)$	$2^{n-1}(2^n-1)$	$G = S$	$n \geq 3$
4	$\mathrm{Sp}_{2n}(2)$	$2^{n-1}(2^n+1)$	$G = S$	$n \geq 3$
5	$\mathrm{PSU}_3(q)$	q^3+1	$\mathrm{PSU}_3(q) \leq G \leq \mathrm{P\Gamma U}_3(q)$	$q \neq 2$
6	$\mathrm{Sz}(q)$	q^2+1	$\mathrm{Sz}(q) \leq G \leq \mathrm{Aut}(\mathrm{Sz}(q))$	$q = 2^{2m+1}$, $m > 0$
7	$\mathrm{Ree}(q)$	q^3+1	$\mathrm{Ree}(q) \leq G \leq \mathrm{Aut}(\mathrm{Ree}(q))$	$q = 3^{2m+1}$, $m > 0$
8	M_n	n	$M_n \leq G \leq \mathrm{Aut}(M_n)$	$n \in \{11,12,22,23,24\}$, M_n Mathieu group, $G = S$ or $n = 22$
9	M_{11}	12	$G = S$	
10	$\mathrm{PSL}_2(11)$	11	$G = S$	
11	$\mathrm{Alt}(7)$	15	$G = S$	
12	$\mathrm{PSL}_2(8)$	28	$G = \mathrm{P\Sigma L}_2(8)$	
13	HS	176	$G = S$	HS Higman-Sims group
14	Co_3	276	$G = S$	Co_3 third Conway group

Every 2-transitive group has the EKR-module Property

Theorem (M., Sin)

All 2-transitive groups have the EKR module property.

Two cases:

1. for groups with regular minimal normal subgroup: show the projection of a maximum coclique to any module is the same as the projection of a canonical coclique.
2. for other groups: first show only need to consider minimal group, and then, since these are known, check each one.

Open Question

What other families of groups have EKR property?

Every 2-transitive group has the EKR-module Property

Corollary

For any 2-transitive group, the characteristic vector of any maximum intersecting set is a linear combination of the $v_{i,j}$.

Corollary

For any 2-transitive group, the characteristic vector of any maximum intersecting set has the same inner distribution as $v_{i,j}$.

Open Question

1. In a 2-transitive group are all maximum cocliques either groups or cosets of groups?
2. In a 2-transitive group, can there be a group that is also a maximum coclique that is not isomorphic to the stabilizer of a point?
3. When does a group have non-conjugate subgroups that give the same induced representation?

Strict-EKR for 2-transitive groups

1. $\mathrm{Sym}(n)$ has strict EKR-property.(Cameron and Ku, Godsil and M.)
2. For $\mathrm{PGL}(n, q)$
   - for $n = 2$ has the strict-EKR property (M. and Spiga);
   - for $n \geq 3$ the maximum intersecting sets are either stabilizers of a point or a hyperplane (M. and Spiga, Spiga).
3. $\mathrm{PSL}(2, q)$ has the strict-EKR property (Long, Plaza, Sin, Xiang).
4. $\mathrm{Alt}(n)$ and the Mathieu groups have the strict EKR (Ahmadi, M.).
5. M_{11} on 12 points has strict EKR
6. $\mathrm{PSL}_2(11)$ on 11 and $\mathrm{Alt}(7)$ on 15 do not have strict EKR.

Open Question

Which 2-transitive groups have the strict EKR property?

1-Transitive Groups

Open Question

Let G be a 1-transitive group. When is it possible to weight the conjugacy classes so that the ratio bound holds with equality?

Set this up as a linear programming problem,

- Put a weight on the conjugacy classes of derangements
- maximize the eigenvalue from the trivial character,
- while keeping all other eigenvalues above -1.
- Also can set the non-trivial representations in the permutation representation to be -1.

General Linear Group $\mathrm{GL}(2, q)$

The group $\mathrm{GL}(2, q)$ acts on the $q^2 - 1$ non-zero vectors in $\mathbb{F}_q^2$.

1. This action is 1-transitive (not 2-transitive).
2. This group has a clique of size $q^2 - 1$.
3. There is a weighting on the conjugacy classes so that the ratio bound holds with equality.
4. With an extra argument (using the clique) $GL(2, q)$ has the EKR-module property.

Theorem (Ahanjideh and Ahanjideh)

$\mathrm{GL}(2, q)$ *has the EKR property, the maximum cocliques are cosets of either the stabilizer of a point or the stabilizer of a line.*

Open Question

What about other actions of $\mathrm{GL}(2, q)$? These should be easy to check because the character table is not so tricky.

t-Intersecting Permutations

$\mathrm{Sym}(n)$ acting on ordered t-sets—or $\mathrm{Sym}(n)/\,\mathrm{Sym}(n-t)$.

Theorem (Ellis, Friedgut, Pilpel 2010)

For n sufficiently large, the maximum t-intersecting set of permutations has size $(n-t)!$ and is the coset of the point-wise stabilizer of a t-set.

$\mathrm{Sym}(n)$ acting on unordered t-sets—or $\mathrm{Sym}(n)/(\mathrm{Sym}(t)\times\mathrm{Sym}(n-t))$.

Theorem (Ellis, 2011)

For n sufficiently large, the maximum t-set-wise intersecting set has size $(n-t)!\,t!$ and is the coset of the stabilizer of a t-set.

Open Question

What is the exact lower bound on n?
What are the maximum sets for small values of n?

A non-EKR group

Example

The group $\mathrm{Sym}(8)$ acting on the ordered 4-sets from $\{1, \ldots, 8\}$.
The set of all permutations that fix at least 5 of [1..6] is intersecting and bigger.

- Subgroup that fixes the elements $\{1, 2, 3, 4\}$ has size $4! = 24$.
- The set that fixes at least 5 of $\{1, 2, 3, 4, 5, 6\}$

$$\underbrace{\binom{6}{6} 2}_{\text{Fix all 6 elements.}} + \underbrace{\binom{6}{5}}_{\text{Pick 5 fixed elements.}} \underbrace{(2)}_{\text{Non-fixed to 7 or 8.}} \underbrace{(2)}_{\text{Place 7 and 8.}} = 26$$

Open Question

Are these sets the largest intersecting sets for some *n*? (How could we prove this?)

Open Question

Is there a similar maximum intersecting set for other groups?

Other Properties

Other researchers have suggested other EKR properties:

- Li, Song, Pantagi: Consider intersecting **groups**.
- Bardestani and Mallahi-Karai: Defined a group **action** to have EKR or Strict EKR property. Then said a **group** has an EKR property if for every subgroup H the group action G in G/H has the EKR property.

Open Question

For any group, what are the maximum intersecting groups?

Open Question

Which groups have the EKR property or the strict EKR property?

Other Problems

Open Question

For 2-transitive groups, what are the boolean vectors in the span of vectors $v_{i,j}$?

Open Question

Which 1-transitive groups have “interesting” intersecting set of permutations?

Open Question

In a transitive group what is the largest set of permutations that is closed under taking conjugation?