Open problems related to balancedly splittable orthogonal designs

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Open Problems in Algebraic Combinatorics
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Hadamard matrix

Definition

An $n \times n$ (± 1)-matrix H is a *Hadamard matrix* if $HH^T = nI$. H(n) denotes a Hadamard matrix of order n.

If there is an H(n), then $n = 1, 2$ or $4k$, k a positive integer.

The BIG open problem:

Conjecture 1: There is a Hadamard matrix of order $4n$ for each natural number n.

Conjecture 1 is confirmed for $n < 167$.

The small open problem 1:

OP1: There is a Hadamard matrix of order $4(167) = 668$.

Comments on OP1

Turyn-type sequences, TT(n), are quadruples of $(-1,1)$-sequences $(A;B;C;D)$, with lengths $(n,n,n,n-1)$ respectively, where the sum of the *non-periodic autocorrelation* functions of A,B and twice that of C,D is a delta-function (i.e., vanishes everywhere except at 0).

Turyn-type sequences TT(n) lead to $H(12n-4)$.

Turyn-type sequences TT(36) led to $H(428)$.

The existence of TT(56) would lead to $H(668)$

Refer to: https://www.cs.uleth.ca/ hadi/research/KlaTurTyp.pdf

Balancedly splittable Hadamard matrices

Definition

A Hadamard matrix H of order n is *balancedly splittable* with the parameters (n, ℓ, a) if by suitably permuting its rows (columns) it can be transformed to

$$H = \begin{bmatrix} H_2 \\ H_1 \end{bmatrix}, (H = \begin{bmatrix} K_2 | K_1 \end{bmatrix})$$

such that H_1 (K_1) is an $\ell \times n$ $(n \times \ell)$ matrix and all off-diagonal entries of $H_1^t H_1$ $(K_1 {K_1}^t)$ belong to the set $\{a, -a\}$, for some positive integer a.

OP2: Is there is a balancedly splittable Hadamard matrix of order 144?

Refer to: https://arxiv.org/abs/2103.04571

Orthogonal designs

Definition

An *orthogonal design* of *order* n and *type* $(s_1, \ldots, s_t)$ in indeterminants $x_1, \ldots, x_t$ is a matrix $D = [d_{ij}]$, $d_{ij} \in \{0, x_1, \ldots, x_t\}$ Such that $DD^t = (sum_{i=1}^{i=t} s_i x_i^2) I_n$.

Number of variables in a *any orthogonal design* is restricted to the *Radon number*.

Definition

The *Radon function*, ρ, is defined by $\rho(n) := 8q + 2^r$ when $n = 2^k \cdot p$, where positive integer p is odd, $k = 4q + r$, and $0 \leq r < 4$. For odd p, $\rho(2^k p)$ depends only on k.

OP3: Is there an OD$(128; 8_{16})$?

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OP4: Is there a balancedly splittable Hadamard matrix of order 144?

Refer to: https://arxiv.org/abs/1806.00165

Balancedly splittable Hadamard matrices and MU Hadamard matrices

Theorem

Let $H = \begin{bmatrix} H_2 \\ H_1 \end{bmatrix}$ *be a balancedly splittable Hadamard matrix with the parameters* (n, ℓ, a)*. Then the following are equivalent.*

- $K = \frac{1}{2a}(H_1^t H_1 - H_2^t H_2)$ *is a Hadamard matrix.*
- $(\ell, a) = (\frac{n \pm \sqrt{n}}{2}, \frac{\sqrt{n}}{2})$.

In this case, $n = 4k^2$ *for some integer* k*, and*

$$HK^t = \sqrt{n} \begin{bmatrix} H_1 \\ -H_2 \end{bmatrix},$$

and thus the Hadamard matrices H *and* K *are* **unbiased***.*

OP5: What is the largest number of Mutually Unbiased Hadamard matrices which is obtained from all balancedly splittable Hadamard matrices of the same order?

Regular Hadamard matrices

A Hadamard matrix of order n is said to be *regular* if the row sums are all the same and equal to $\sqrt{n}$. In this case n must be square. A *Menon design* is a SBIBD$(4m^2, 2m^2 \pm m, m^2 \pm m)$.

A *Graphical Hadamard matrix* is a symmetric Hadamard matrix with constant diagonal. It is known that regular Graphical Hadamard matrices exist of order $4n^4$ for any natural number n.

OP6: Is there a Menon design SBIBD$(4m^2, 2m^2 \pm m, m^2 \pm m)$ for each natural number m?

Refer to: Muzychuk, Mikhail; Xiang, Qing, Symmetric Bush-type Hadamard matrices of order $4m^4$ exist for all odd m. Proc. Amer. Math. Soc. 134 (2006), no. 8, 2197–2204.