# Balancedly splittable orthogonal designs

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Open Problems in Algebraic Combinatorics
Joint work with Thomas Pender and Sho Suda

# Hadamard matrix

**Definition**

An $n \times n$ $(\pm 1)$-matrix $H$ is a *Hadamard matrix* if $HH^T = nI$ (i.e., its rows are pairwise orthogonal). H$(n)$ denotes a Hadamard matrix of order $n$.

If there is an H$(n)$, then $n = 1, 2$ or $4k$, $k$ a positive integer.

The BIG open problem:

**Conjecture 1: There is a Hadamard matrix of order $4n$ for each natural number $n$.**

Conjecture 1 is confirmed for $n < 167$.

The small open problem 1:

**OP1: There is a Hadamard matrix of order $4(167) = 668$.**

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*Turyn-type sequences*, $\mathsf{TT}(n)$, are quadruples of $(-1,1)$-sequences $(A;B;C;D)$, with lengths $(n,n,n,n-1)$ respectively, where the sum of the *non-periodic autocorrelation* functions of $A,B$ and twice that of $C,D$ is a delta-function (i.e., vanishes everywhere except at 0).

**Turyn-type sequences TT$(n)$ lead to $H(12n-4)$.**

Turyn-type sequences $\mathsf{TT}(36)$ led to $H(428)$.

The existence of $\mathsf{TT}(56)$ would lead to $H(668)$

Caution!

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# Halving a Hadamard matrix

Here is an example of a *halved* Hadamard matrix of order 16:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
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1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
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1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
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1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
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1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
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# Equiangular frames

Here is an example of a *frame* in $\mathbb{R}^6$

$$F_1 = \left[\begin{array}{cccc|cccccc|cccccc} - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\ - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\ - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\ - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\ - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\ - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 \end{array}\right]$$

Considering each column of $F_1$ as a vector in $\mathbb{R}^6$. Let $\mathbf{u}_i$ be the $i$-th column of $F_1$ and the line $l_i$ containing $\mathbf{u}_i$. Then the lines $l_i$, $1 \leq i \leq 16$ form an *Equiangular lines* set in $\mathbb{R}^6$, noting that:

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$$F_1^T F_1 = \begin{bmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 6 & 2 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6
\end{bmatrix}$$

$\bar{x} = -x$

# Equiangular Tight Frame, ETF

The Equiangular Fame $F_1$ is *Tight* if the rows of $F_1$ are pairwise orthogonal, i.e. $F_1 F_1^T = 16 I_{16}$

A Frame is called *Flat* if all the entries are of equal absolute values.
A second flat ETF in $\mathbb{R}^6$:

$$F_2 = \left[\begin{array}{cccc|cccccc|cccccc} 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\ 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\ 1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\ 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\ 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\ 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \end{array}\right]$$

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A *balancedly split Hadamard* matrix of order 16 containing $F_1$ and $F_2$ as submatrices:

$$
\left[\begin{array}{cccc|cccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
\hline
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
\hline
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
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1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
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1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
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A Hadamard matrix $H$ of order $n$ is *balancedly splittable* with the parameters $(n, \ell, a)$ if by suitably permuting its rows (columns) it can be transformed to

$$H = \begin{bmatrix} H_2 \\ H_1 \end{bmatrix}, (H = [K_2|K_1])$$

such that $H_1$ $(K_1)$ is an $\ell \times n$ $(n \times \ell)$ matrix and all off-diagonal entries of $H_1^t H_1$ $(K_1 {K_1}^t)$ belong to the set $\{a, -a\}$, for some positive integer $a$.

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## Theorem

*Let* $H = \begin{bmatrix} H_2 \\ H_1 \end{bmatrix}$ *be a balancedly splittable Hadamard matrix with the parameters* $(n, \ell, a)$*. Then the following are equivalent.*

- $K = \frac{1}{2a}(H_1^t H_1 - H_2^t H_2)$ *is a Hadamard matrix.*
- $(\ell, a) = (\frac{n \pm \sqrt{n}}{2}, \frac{\sqrt{n}}{2})$.

*In this case,* $n = 4k^2$ *for some integer* $k$*, and*

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A Hadamard matrix of order $n$ is said to be *regular* if the row sums are all the same and equal to $\sqrt{n}$. In this case $n$ must be square.

**Lemma**

*Any balancedly splittable Hadamard matrix with the parameters $(4n^2, \ell, a) = (4n^2, 2n^2 - n, n)$ is* **equivalent** *to a regular Hadamard matrix.*

Two of the five Hadamard matrices of order 16 fail to be balancedly splittable.

**Lemma**

*There is no balancedly splittable Hadamard matrix with parameters $(4n^2, 2n^2 - n, n)$, $n$ an odd integer.*

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# Orthogonal designs

**Definition**

Let $n$ and $w$ be positive integers. A *weighing matrix* $W$, of *weight* $w$ and *order* $n$, is an $n \times n$ $(0, \pm 1)$-matrix satisfying $WW^T = wI$. We denote such a matrix by $W(n, w)$. ($w = 0$ is generally not permitted.)

**Definition**

An *orthogonal design* of *order* $n$ and *type* $(s_1, \ldots, s_u)$ (or with *parameters* $s_1, \ldots, s_u$), is a matrix of the form $D = x_1 W_1 + x_2 W_2 + \cdots + x_u W_u$, where $x_1, \ldots, x_u$ are distinct commuting indeterminates, $W_i = W(n, s_i)$, $i = 1, \ldots, u$, and $W_i$ and $W_j$ are disjoint and antiamicable, for all $1 \leq i < j \leq u$. Such a design is denoted $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$. It is *full* if $\sum_{i=1}^{u} s_i = n$.

# Orthogonal designs

## Definition

Let $n$ and $w$ be positive integers. A *weighing matrix* $W$, of *weight* $w$ and *order* $n$, is an $n \times n$ $(0, \pm 1)$-matrix satisfying $WW^T = wI$. We denote such a matrix by $W(n, w)$. ($w = 0$ is generally not permitted.)

## Definition

An *orthogonal design* of *order* $n$ and *type* $(s_1, \ldots, s_u)$ (or with *parameters* $s_1, \ldots, s_u$), is a matrix of the form $D = x_1W_1 + x_2W_2 + \cdots + x_uW_u$, where $x_1, \ldots, x_u$ are distinct commuting indeterminates, $W_i = W(n, s_i)$, $i = 1, \ldots, u$, and $W_i$ and $W_j$ are disjoint and antiamicable, for all $1 \leq i < j \leq u$. Such a design is denoted $\text{OD}(n; s_1, s_2, \ldots, s_u)$. It is *full* if $\sum_{i=1}^{u} s_i = n$.

# Example of orthogonal designs

$$D_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad D_3 = \begin{pmatrix} d & a & b & c \\ -a & d & c & -b \\ -b & -c & d & a \\ -c & b & -a & d \end{pmatrix}$$

OD(2;1,1) OD(4;1,1,1,1)

Here is an $OD(6;1)$, also a $W(6,5)$: $a \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{pmatrix}$.

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OD(2;1,1)

$$D_3 = \begin{pmatrix} d & a & b & c \\ -a & d & c & -b \\ -b & -c & d & a \\ -c & b & -a & d \end{pmatrix}$$

OD(4;1,1,1,1)

Here is an $OD(6;1)$, also a $W(6,5)$: $a \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{pmatrix}$.

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# More example of orthogonal designs

Note that $\mathbf{x} = -x$.

$$\begin{pmatrix} a & b & d & c & f & e & g & h \\ \mathbf{b} & a & c & \mathbf{d} & e & \mathbf{f} & h & \mathbf{g} \\ \mathbf{d} & \mathbf{c} & a & b & \mathbf{g} & h & f & \mathbf{e} \\ \mathbf{c} & d & \mathbf{b} & a & h & g & \mathbf{e} & \mathbf{f} \\ \mathbf{f} & \mathbf{e} & g & \mathbf{h} & a & b & \mathbf{d} & c \\ \mathbf{e} & f & \mathbf{h} & \mathbf{g} & \mathbf{b} & a & c & d \\ \mathbf{g} & \mathbf{h} & \mathbf{f} & e & d & \mathbf{c} & a & b \\ \mathbf{h} & g & e & f & \mathbf{c} & \mathbf{d} & \mathbf{b} & a \end{pmatrix}$$

an OD(8;1,1,1,1,1,1,1,1)

$$\begin{pmatrix} a & b & c & b & \mathbf{a} & d & a & d & \mathbf{c} & d & c & \mathbf{b} \\ c & a & b & \mathbf{a} & d & b & d & \mathbf{c} & a & c & \mathbf{b} & d \\ b & c & a & d & b & \mathbf{a} & \mathbf{c} & a & d & \mathbf{b} & d & c \\ \mathbf{b} & a & \mathbf{d} & a & b & c & c & d & \mathbf{b} & \mathbf{d} & \mathbf{a} & c \\ a & \mathbf{d} & \mathbf{b} & c & a & b & d & \mathbf{b} & c & \mathbf{a} & c & \mathbf{d} \\ \mathbf{d} & \mathbf{b} & a & b & c & a & \mathbf{b} & c & d & c & \mathbf{d} & \mathbf{a} \\ \mathbf{a} & \mathbf{d} & c & \mathbf{c} & \mathbf{d} & b & a & b & c & \mathbf{a} & b & d \\ \mathbf{d} & c & \mathbf{a} & \mathbf{d} & b & \mathbf{c} & c & a & b & b & d & \mathbf{a} \\ c & \mathbf{a} & \mathbf{d} & b & \mathbf{c} & \mathbf{d} & b & c & a & d & \mathbf{a} & b \\ \mathbf{d} & \mathbf{c} & b & d & a & \mathbf{c} & a & \mathbf{b} & \mathbf{d} & a & b & c \\ \mathbf{c} & b & \mathbf{d} & a & \mathbf{c} & d & \mathbf{b} & \mathbf{d} & a & c & a & b \\ b & \mathbf{d} & \mathbf{c} & \mathbf{c} & d & a & \mathbf{d} & a & \mathbf{b} & b & c & a \end{pmatrix}.$$

an OD(12;3,3,3,3)

# More example of orthogonal designs

Note that $\mathbf{x} = -x$.

$$
\begin{pmatrix}
a & b & d & c & f & e & g & h \\
\mathbf{b} & a & c & \mathbf{d} & e & \mathbf{f} & h & \mathbf{g} \\
\mathbf{d} & \mathbf{c} & a & b & \mathbf{g} & h & f & \mathbf{e} \\
\mathbf{c} & d & \mathbf{b} & a & h & g & \mathbf{e} & \mathbf{f} \\
\mathbf{f} & \mathbf{e} & g & \mathbf{h} & a & b & \mathbf{d} & c \\
\mathbf{e} & f & \mathbf{h} & \mathbf{g} & \mathbf{b} & a & c & d \\
\mathbf{g} & \mathbf{h} & \mathbf{f} & e & d & \mathbf{c} & a & b \\
\mathbf{h} & g & e & f & \mathbf{c} & \mathbf{d} & \mathbf{b} & a
\end{pmatrix}
\quad
\begin{pmatrix}
a & b & c & b & \mathbf{a} & d & a & d & \mathbf{c} & d & c & \mathbf{b} \\
c & a & b & \mathbf{a} & d & b & d & \mathbf{c} & a & c & \mathbf{b} & d \\
b & c & a & d & b & \mathbf{a} & \mathbf{c} & a & d & \mathbf{b} & d & c \\
\mathbf{b} & a & \mathbf{d} & a & b & c & c & d & \mathbf{b} & \mathbf{d} & \mathbf{a} & c \\
a & \mathbf{d} & \mathbf{b} & c & a & b & d & \mathbf{b} & c & \mathbf{a} & c & \mathbf{d} \\
\mathbf{d} & \mathbf{b} & a & b & c & a & \mathbf{b} & c & d & c & \mathbf{d} & \mathbf{a} \\
\mathbf{a} & \mathbf{d} & c & \mathbf{c} & \mathbf{d} & b & a & b & c & \mathbf{a} & b & d \\
\mathbf{d} & c & \mathbf{a} & \mathbf{d} & b & \mathbf{c} & c & a & b & b & d & \mathbf{a} \\
c & \mathbf{a} & \mathbf{d} & b & \mathbf{c} & \mathbf{d} & b & c & a & d & \mathbf{a} & b \\
\mathbf{d} & \mathbf{c} & b & d & a & \mathbf{c} & a & \mathbf{b} & \mathbf{d} & a & b & c \\
\mathbf{c} & b & \mathbf{d} & a & \mathbf{c} & d & \mathbf{b} & \mathbf{d} & a & c & a & b \\
b & \mathbf{d} & \mathbf{c} & \mathbf{c} & d & a & \mathbf{d} & a & \mathbf{b} & b & c & a
\end{pmatrix}.
$$

an OD(8;1,1,1,1,1,1,1,1)

an OD(12;3,3,3,3)

# More example of orthogonal designs

Note that $\mathbf{x} = -x$.

$$\begin{pmatrix} a & b & d & c & f & e & g & h \\ \mathbf{b} & a & c & \mathbf{d} & e & \mathbf{f} & h & \mathbf{g} \\ \mathbf{d} & \mathbf{c} & a & b & \mathbf{g} & h & f & \mathbf{e} \\ \mathbf{c} & d & \mathbf{b} & a & h & g & \mathbf{e} & \mathbf{f} \\ \mathbf{f} & \mathbf{e} & g & \mathbf{h} & a & b & \mathbf{d} & c \\ \mathbf{e} & f & \mathbf{h} & \mathbf{g} & \mathbf{b} & a & c & d \\ \mathbf{g} & \mathbf{h} & \mathbf{f} & e & d & \mathbf{c} & a & b \\ \mathbf{h} & g & e & f & \mathbf{c} & \mathbf{d} & \mathbf{b} & a \end{pmatrix}$$

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$$\begin{pmatrix} a & b & c & b & \mathbf{a} & d & a & d & \mathbf{c} & d & c & \mathbf{b} \\ c & a & b & \mathbf{a} & d & b & d & \mathbf{c} & a & c & \mathbf{b} & d \\ b & c & a & d & b & \mathbf{a} & \mathbf{c} & a & d & \mathbf{b} & d & c \\ \mathbf{b} & a & \mathbf{d} & a & b & c & c & d & \mathbf{b} & \mathbf{d} & \mathbf{a} & c \\ a & \mathbf{d} & \mathbf{b} & c & a & b & d & \mathbf{b} & c & \mathbf{a} & c & \mathbf{d} \\ \mathbf{d} & \mathbf{b} & a & b & c & a & \mathbf{b} & c & d & c & \mathbf{d} & \mathbf{a} \\ \mathbf{a} & \mathbf{d} & c & \mathbf{c} & \mathbf{d} & b & a & b & c & \mathbf{a} & b & d \\ \mathbf{d} & c & \mathbf{a} & \mathbf{d} & b & \mathbf{c} & c & a & b & b & d & \mathbf{a} \\ c & \mathbf{a} & \mathbf{d} & b & \mathbf{c} & \mathbf{d} & b & c & a & d & \mathbf{a} & b \\ \mathbf{d} & \mathbf{c} & b & d & a & \mathbf{c} & a & \mathbf{b} & \mathbf{d} & a & b & c \\ \mathbf{c} & b & \mathbf{d} & a & \mathbf{c} & d & \mathbf{b} & \mathbf{d} & a & c & a & b \\ b & \mathbf{d} & \mathbf{c} & \mathbf{c} & d & a & \mathbf{d} & a & \mathbf{b} & b & c & a \end{pmatrix}.$$

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\mathbf{d} & \mathbf{c} & a & b & \mathbf{g} & h & f & \mathbf{e} \\
\mathbf{c} & d & \mathbf{b} & a & h & g & \mathbf{e} & \mathbf{f} \\
\mathbf{f} & \mathbf{e} & g & \mathbf{h} & a & b & \mathbf{d} & c \\
\mathbf{e} & f & \mathbf{h} & \mathbf{g} & \mathbf{b} & a & c & d \\
\mathbf{g} & \mathbf{h} & \mathbf{f} & e & d & \mathbf{c} & a & b \\
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\end{pmatrix}
\quad
\begin{pmatrix}
a & b & c & b & \mathbf{a} & d & a & d & \mathbf{c} & d & c & \mathbf{b} \\
c & a & b & \mathbf{a} & d & b & d & \mathbf{c} & a & c & \mathbf{b} & d \\
b & c & a & d & b & \mathbf{a} & \mathbf{c} & a & d & \mathbf{b} & d & c \\
\mathbf{b} & a & \mathbf{d} & a & b & c & c & d & \mathbf{b} & \mathbf{d} & \mathbf{a} & c \\
a & \mathbf{d} & \mathbf{b} & c & a & b & d & \mathbf{b} & c & \mathbf{a} & c & \mathbf{d} \\
\mathbf{d} & \mathbf{b} & a & b & c & a & \mathbf{b} & c & d & c & \mathbf{d} & \mathbf{a} \\
\mathbf{a} & \mathbf{d} & c & \mathbf{c} & \mathbf{d} & b & a & b & c & \mathbf{a} & b & d \\
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an OD(8;1,1,1,1,1,1,1,1) — an OD(12;3,3,3,3)

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$$
\begin{pmatrix}
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\mathbf{g} & \mathbf{h} & \mathbf{f} & e & d & \mathbf{c} & a & b \\
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\mathbf{b} & a & \mathbf{d} & a & b & c & c & d & \mathbf{b} & \mathbf{d} & \mathbf{a} & c \\
a & \mathbf{d} & \mathbf{b} & c & a & b & d & \mathbf{b} & c & \mathbf{a} & c & \mathbf{d} \\
\mathbf{d} & \mathbf{b} & a & b & c & a & \mathbf{b} & c & d & c & \mathbf{d} & \mathbf{a} \\
\mathbf{a} & \mathbf{d} & c & \mathbf{c} & \mathbf{d} & b & a & b & c & \mathbf{a} & b & d \\
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\mathbf{c} & b & \mathbf{d} & a & \mathbf{c} & d & \mathbf{b} & \mathbf{d} & a & c & a & b \\
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\end{pmatrix}.
$$

an OD(8;1,1,1,1,1,1,1,1) an OD(12;3,3,3,3)

# Open Problem 2

There are many open questions related to orthogonal designs and most are elusive.

Number of variables in a *full orthogonal design* (no zero entries) is restricted to the *Radon number*.

**Definition**

The *Radon function*, $\rho$, is defined by $\rho(n) := 8q + 2^r$ when $n = 2^k \cdot p$, where positive integer $p$ is odd, $k = 4q + r$, and $0 \leq r < 4$. For odd $p$, $\rho(2^k p)$ depends only on $k$.

The first few values of $\rho(2^k)$:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho(2^k)$ | 1 | 2 | 4 | 8 | 9 | 10 | 12 | 16 | 17 | 18 | 20 | 24 | 25 | 26 | 28 | 32 |

**OP2: Is there an OD$(128; 8_{16})$?**

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho(2^k)$ | 1 | 2 | 4 | 8 | 9 | 10 | 12 | 16 | 17 | 18 | 20 | 24 | 25 | 26 | 28 | 32 |

**OP2: Is there an OD$(128; 8_{16})$?**

# Halving an orthogonal design

Here is a halved *orthogonal design* of order 16 in two variables $a$ and $b$.

$$
\begin{bmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b
\end{bmatrix}
$$

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$$
\begin{bmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b
\end{bmatrix}
$$

# Halving an orthogonal design

Here is a halved *orthogonal design* of order 16 in two variables $a$ and $b$.

$$
\begin{bmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b
\end{bmatrix}
$$

Transpose of first vertical frame.

$$F_1^t = \begin{bmatrix} a\,\bar{a}\,a\,\bar{a}\,a\,a\,b\,\bar{b}\,b\,\bar{b}\,a\,a\,\bar{b}\,b\,b\,\bar{b} \\ b\,\bar{b}\,b\,\bar{b}\,b\,b\,\bar{a}\,a\,\bar{a}\,a\,b\,b\,a\,\bar{a}\,\bar{a}\,a \\ a\,a\,\bar{a}\,\bar{a}\,b\,\bar{b}\,a\,a\,b\,\bar{b}\,b\,\bar{b}\,a\,a\,\bar{b}\,b \\ b\,b\,\bar{b}\,\bar{b}\,\bar{a}\,a\,b\,b\,\bar{a}\,a\,\bar{a}\,a\,b\,b\,a\,\bar{a} \\ a\,\bar{a}\,\bar{a}\,a\,b\,\bar{b}\,b\,\bar{b}\,a\,a\,\bar{b}\,b\,b\,\bar{b}\,a\,a \\ b\,\bar{b}\,\bar{b}\,b\,\bar{a}\,a\,\bar{a}\,a\,b\,b\,a\,\bar{a}\,\bar{a}\,a\,b\,b \end{bmatrix}$$

Transpose of second vertical frame.

$$F_2^t = \begin{bmatrix} \bar{a}\,a\,\bar{a}\,a\,a\,a\,\bar{b}\,b\,b\,\bar{b}\,a\,a\,b\,\bar{b}\,b\,\bar{b} \\ \bar{b}\,b\,\bar{b}\,b\,b\,b\,a\,\bar{a}\,\bar{a}\,a\,b\,b\,\bar{a}\,a\,\bar{a}\,a \\ \bar{a}\,\bar{a}\,a\,a\,b\,\bar{b}\,a\,a\,\bar{b}\,b\,b\,\bar{b}\,a\,a\,b\,\bar{b} \\ \bar{b}\,\bar{b}\,b\,b\,\bar{a}\,a\,b\,b\,a\,\bar{a}\,\bar{a}\,a\,b\,b\,\bar{a}\,a \\ \bar{a}\,a\,a\,\bar{a}\,\bar{b}\,b\,b\,\bar{b}\,a\,a\,b\,\bar{b}\,b\,\bar{b}\,a\,a \\ \bar{b}\,b\,b\,\bar{b}\,a\,\bar{a}\,\bar{a}\,a\,b\,b\,\bar{a}\,a\,\bar{a}\,a\,b\,b \end{bmatrix}$$

$$F_1 F_1^t = (a^2 + b^2) \begin{bmatrix} 3 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ - & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - \\ - & - & 3 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & - & - \\ - & - & - & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & 1 & - & 3 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\ 1 & - & 1 & - & - & 3 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\ 1 & 1 & - & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1 & 1 & - \\ 1 & 1 & - & - & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1 & - & 1 \\ 1 & - & - & 1 & 1 & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1 \\ 1 & - & - & 1 & - & 1 & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1 \\ 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 3 & - & - & 1 & - & 1 \\ 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 3 & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & - & - & 1 \\ 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3 & 1 & - \\ 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & - \\ 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3 \end{bmatrix}$$

and

$$F_2 F_2^t = (a^2 + b^2) \begin{bmatrix} 3 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & 3 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\ - & - & 3 & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\ - & - & - & 3 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - \\ - & 1 & - & 1 & 3 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\ - & 1 & - & 1 & - & 3 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\ - & - & 1 & 1 & - & 1 & 3 & - & - & 1 & 1 & - & 1 & 1 & - & 1 \\ - & - & 1 & 1 & 1 & - & - & 3 & 1 & - & - & 1 & 1 & 1 & 1 & - \\ - & 1 & 1 & - & - & 1 & - & 1 & 3 & - & - & 1 & 1 & - & 1 & 1 \\ - & 1 & 1 & - & 1 & - & 1 & - & - & 3 & 1 & - & - & 1 & 1 & 1 \\ - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 3 & - & 1 & - & 1 & - \\ - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & 3 & - & 1 & - & 1 \\ - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 3 & - & 1 & - \\ - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 3 & - & 1 \\ - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 3 & - \\ - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & 3 \end{bmatrix}$$

# Balancedly splittable orthogonal designs

**Definition**

The orthogonal design $X_n$ is said to be *balancedly splittable* if $X_n$ contains an $m \times n$ submatrix $X_1$ such that all of the off diagonal entries of $X_1^* X_1$ are in the set

$$\{\pm c\sigma \mid \sigma = \sum_{\ell=1}^{u} s_\ell |x_\ell|^2\},$$

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# Balancedly splittable orthogonal designs

$$
\left[\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
\hline
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b
\end{array}\right]
$$

# Balancedly splittable orthogonal designs

$$
\left[\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
\hline
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b
\end{array}\right]
$$

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$$
\left[\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
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b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
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\end{array}\right]
$$

# Construction method

**Theorem**

*If there is an OD$(n; s_1, \ldots, s_u)$, then there is a balancedly splittable OD$(4n^2; 4ns_1, 4ns_2, \ldots, 4ns_u)$.*

There are nine submatrices forming the desired matrix as follows:

$$\begin{bmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{bmatrix}.$$

To form the five submatrices $A, B, E, F, G$ we start with an OD$(2; 1, 1)$ with commuting real variables $a, b$.Let

$$X_2 = \begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix}$$

and

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

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and

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# Construction method

**Theorem**

*If there is an OD*$(n; s_1, \ldots, s_u)$*, then there is a balancedly splittable OD*$(4n^2; 4ns_1, 4ns_2, \ldots, 4ns_u)$.

There are nine submatrices forming the desired matrix as follows:

$$\begin{bmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{bmatrix}.$$

To form the five submatrices $A, B, E, F, G$ we start with an OD$(2; 1, 1)$ with commuting real variables $a, b$.Let

$$X_2 = \begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix}$$

and

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}.$$

The two *auxiliary matrices* $C_1$ and $C_2$ corresponding to $X_2$ are defined by

$$C_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$$

,

$$C_2 = \begin{bmatrix} 1 \\ - \end{bmatrix} \begin{bmatrix} b & \bar{a} \end{bmatrix} = \begin{bmatrix} b & \bar{a} \\ \bar{b} & a \end{bmatrix}.$$

The four rows of $H_2 \otimes X_2$ is used in forming $E$, $G$, and $F$.

The block sequences $(C_1, C_2, C_2)$ and $(C_1, C_2, -C_2)$ forms a *block Golay pair*. Let

$$A = circ(C_1 C_2 C_2) = \begin{bmatrix} C_1 & C_2 & C_2 \\ C_2 & C_1 & C_2 \\ C_2 & C_2 & C_1 \end{bmatrix}$$

$$B = circ(C_1 C_2 \bar{C}_2) = \begin{bmatrix} C_1 & C_2 & \bar{C}_2 \\ \bar{C}_2 & C_1 & C_2 \\ C_2 & \bar{C}_2 & C_1 \end{bmatrix}.$$

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# Construction continues

We get:

$$E = \left[\begin{array}{cccc} b & \bar{a} & b & \bar{a} \\ b & \bar{a} & b & \bar{a} \\ \hline a & b & \bar{a} & \bar{b} \\ a & b & \bar{a} & \bar{b} \\ \hline b & \bar{a} & \bar{b} & a \\ b & \bar{a} & \bar{b} & a \end{array}\right] \qquad A = \left[\begin{array}{cc|cc|cc} a & b & b & \bar{a} & b & \bar{a} \\ a & b & \bar{b} & a & \bar{b} & a \\ \hline b & \bar{a} & a & b & b & \bar{a} \\ \bar{b} & a & a & b & \bar{b} & a \\ \hline b & \bar{a} & b & \bar{a} & a & b \\ \bar{b} & a & \bar{b} & a & a & b \end{array}\right] \qquad B = \left[\begin{array}{cc|cc|cc} a & b & b & \bar{a} & \bar{b} & a \\ a & b & \bar{b} & a & b & \bar{a} \\ \hline \bar{b} & a & a & b & b & \bar{a} \\ b & \bar{a} & a & b & \bar{b} & a \\ \hline b & \bar{a} & \bar{b} & a & a & b \\ \bar{b} & a & b & \bar{a} & a & b \end{array}\right]$$

The first horizontal frame is now constructed:

$$\begin{bmatrix} E & A & B \end{bmatrix}$$

The second horizontal frame is at hand too:

$$\begin{bmatrix} -E & B & A \end{bmatrix}$$

# Construction continues

We get:

$$E = \left[\begin{array}{cccc} b & \bar{a} & b & \bar{a} \\ b & \bar{a} & b & \bar{a} \\ \hline a & b & \bar{a} & \bar{b} \\ a & b & \bar{a} & \bar{b} \\ \hline b & \bar{a} & \bar{b} & a \\ b & \bar{a} & \bar{b} & a \end{array}\right] \qquad A = \left[\begin{array}{cc|cc|cc} a & b & b & \bar{a} & b & \bar{a} \\ a & b & \bar{b} & a & \bar{b} & a \\ \hline b & \bar{a} & a & b & b & \bar{a} \\ \bar{b} & a & a & b & \bar{b} & a \\ \hline b & \bar{a} & b & \bar{a} & a & b \\ \bar{b} & a & \bar{b} & a & a & b \end{array}\right] \qquad B = \left[\begin{array}{cc|cc|cc} a & b & b & \bar{a} & \bar{b} & a \\ a & b & \bar{b} & a & b & \bar{a} \\ \hline \bar{b} & a & a & b & b & \bar{a} \\ b & \bar{a} & a & b & \bar{b} & a \\ \hline b & \bar{a} & \bar{b} & a & a & b \\ \bar{b} & a & b & \bar{a} & a & b \end{array}\right]$$

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The second horizontal frame is at hand too:

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It remains to complement the design by the following two matrices.

$$G = \begin{bmatrix} a\,b\,a\,b \\ a\,b\,a\,b \\ a\,b\,a\,b \\ a\,b\,a\,b \end{bmatrix} \qquad F = \left[\begin{array}{c|c|c} a\,b & a\,b & a\,b \\ \bar{a}\,\bar{b} & a\,b & \bar{a}\,\bar{b} \\ a\,b & \bar{a}\,\bar{b} & \bar{a}\,\bar{b} \\ \bar{a}\,\bar{b} & \bar{a}\,\bar{b} & a\,b \end{array}\right]$$

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$$G = \begin{bmatrix} a\,b\,a\,b \\ a\,b\,a\,b \\ a\,b\,a\,b \\ a\,b\,a\,b \end{bmatrix} \qquad F = \left[\begin{array}{c|c|c} a\;b & a\;b & a\;b \\ \bar{a}\;\bar{b} & a\;b & \bar{a}\;\bar{b} \\ a\;b & \bar{a}\;\bar{b} & \bar{a}\;\bar{b} \\ \bar{a}\;\bar{b} & \bar{a}\;\bar{b} & a\;b \end{array}\right]$$

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$$G = \begin{bmatrix} a\,b\,a\,b \\ a\,b\,a\,b \\ a\,b\,a\,b \\ a\,b\,a\,b \end{bmatrix} \qquad F = \left[\begin{array}{cc|cc|cc} a & b & a & b & a & b \\ \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} \\ a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\ \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b \end{array}\right]$$

Putting these together, we obtain a balancedly splittable OD$(16; 8, 8)$.

$$\left[\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right]$$

Given the COD$(2;1,1)$ $\begin{bmatrix} a & b \\ \bar{b}^* & a^* \end{bmatrix}$, we have the following balancedly split COD$(16;8,8)$

$$\left[\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
\bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* \\
\bar{b}^* & a^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* \\
a & b & \bar{a} & \bar{b} & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* \\
a & b & \bar{a} & \bar{b} & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* \\
\bar{b}^* & a^* & b^* & \bar{a}^* & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b \\
\bar{b}^* & a^* & b^* & \bar{a}^* & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b \\
\hline
b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* \\
b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* \\
\bar{a} & \bar{b} & a & b & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* \\
\bar{a} & \bar{b} & a & b & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* \\
b^* & \bar{a}^* & \bar{b}^* & a^* & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b \\
b^* & \bar{a}^* & \bar{b}^* & a^* & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b
\end{array}\right].$$

Consider the COD$(2;1,1)$ given by $\begin{bmatrix} a & b \\ \bar{b}^* & a^* \end{bmatrix}$. Using the construction, we have the following COD$(16;8,8)$

$$\left[\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
\bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* \\
\bar{b}^* & a^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* \\
a & b & \bar{a} & \bar{b} & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* \\
a & b & \bar{a} & \bar{b} & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* \\
\bar{b}^* & a^* & b^* & \bar{a}^* & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b \\
\bar{b}^* & a^* & b^* & \bar{a}^* & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b \\
\hline
b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* \\
b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* \\
\bar{a} & \bar{b} & a & b & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b & \bar{b}^* & a^* \\
\bar{a} & \bar{b} & a & b & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b & b^* & \bar{a}^* \\
b^* & \bar{a}^* & \bar{b}^* & a^* & \bar{b}^* & a^* & b^* & \bar{a}^* & a & b & \bar{b}^* & a^* & \bar{b}^* & a^* & a & b \\
b^* & \bar{a}^* & \bar{b}^* & a^* & b^* & \bar{a}^* & \bar{b}^* & a^* & a & b & b^* & \bar{a}^* & b^* & \bar{a}^* & a & b
\end{array}\right].$$

Given the QOD$(2;1,1)$ $\begin{bmatrix} \bar{a} & bi \\ \bar{b}j & ak \end{bmatrix}$, we have the following balancedly split QOD$(16;8,8)$

$$
\left[\begin{array}{cccccccccccccccc}
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}I \\
\hline
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k \\
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k \\
\bar{b}j & ak & bj & \bar{a}k & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi \\
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\hline
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak \\
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k \\
a & \bar{b}i & \bar{a} & bi & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak \\
a & \bar{b}i & \bar{a} & bi & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k \\
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# Balancedly splittable Hadamard matrices of order $16n^2$

**Theorem**

*There is a balancedly splittable Hadamard matrix with parameters $(16n^2, 2n(4n-1), 2n)$ for each $n$ for which there is a Hadamard matrix of order $2n$.*

This leaves the existence of balancedly splittable Hadamard matrices of order $16n^2$, $n$ odd, open starting from $n = 3$.

**OP3: There is (no) balancedly splittable Hadamard matrix of order $16n^2$, $n$ odd**.

The first open case is $H(144)$.

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*There is a balancedly splittable quaternary Hadamard matrix with parameters $(16n^2, 2n(4n-1), 2n)$ for each $n$ for which there is a quaternary Hadamard matrix of order $2n$.*

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# An upper bound for Equiangular lines set

Let $X = \{L_!, L_2, \ldots, L_k\}$ be a finite set of lines in $\mathbb{R}^m$ and let the line $L_i$ be spanned by the unit vector $\mathbf{u}_i$. $X$ is said to form an equiangular lines set, if $|\langle \mathbf{u}_i, \mathbf{u}_j \rangle| = \alpha$, for some number $0 < \alpha < 1$, $i \neq j$.

The following upper bound is due to Delsarte, Goethals and Seidel (1975).

Let $X \subset \mathbb{R}^m$ be a set of unit vectors such that $|\langle v, w \rangle| = \alpha$ for all $v, w \in X, v \neq w$. If $m < \frac{1}{\alpha^2}$, then

$$|X| \leq \frac{m(1-\alpha^2)}{1-m\alpha^2}. \quad (1)$$

**Theorem**

*There is a (twin) flat ETF in* $\mathbb{R}^{2n(4n-1)}$ *with* $\alpha = \frac{1}{4n-1}$ *meeting the upper bound* $(1)$ *for each* $n$ *for which there is a Hadamard matrix of order* $2n$.

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# For Hadamard matrices

- There is a balancedly splittable Hadamard matrix of order $4n^2$ for any $n$ an order of a Hadamard matrix. Case of $n = 12$.
- There is a twin set of flat ETF meeting the Delsarte, Goethals and Seidel bound in $\mathbb{R}^{2n^2-n}$ for each $n$ an order of a Hadamard matrix.
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