Variations of the EKR Problem

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Basic Structure

Goal: Sketch some EKR-related problems which I like.

Three Structures:

- **Sets:** k-sets of $[n] := \{1, \ldots, n\}$.
  Two sets A and B intersect if $|A \cap B| \geq 1$.

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  Two subspaces A and B intersect if $\dim(A \cap B) \geq 1$.

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- **Vector Spaces:** k-spaces of $\mathbb{F}_q^n$.
  Two subspaces A and B intersect if $\dim(A \cap B) \geq 1$.
- **Groups:** elements of a permutation group G.
  Two elements σ and τ intersect if $\mathrm{Fix}(\sigma\tau^{-1}) \geq 1$.

Priority will be on **sets**. (. . . as there the problems are well-investigated.)

Disclaimer: My choice of references in each topic is very subjective.

Nice Intersecting Families

A star: All k-sets containing 1.

All k-sets containing **2 of** $\{1, 2, 3\}$.

All k-sets containing $\frac{t+1}{2}$ **of** $[t] = \{1, \ldots, t\}$.

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Definition

A family $\mathcal{F}$ is a t-**junta** if there exists a t-set T such that we can decide $F\in\mathcal{F}$ based on $F\cap T$.

All of the above are **juntas** with $t<k$!

1-junta: dictatorship.

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All of the above are **juntas** with $t < k$! 1-junta: dictatorship.

Non-small-junta example: We have $F \in \mathcal{F}$ if either

1. $1 \in F$ and $|F \cap \{2, \ldots, k+1\}| \geq 1$, or (in 1-junta)
2. $F = \{2, \ldots, k+1\}$. (nonjunta part)

Intersecting Families as (real) Polynomials

Write the characteristic function of a **family** $\mathcal{F} \subseteq \binom{[n]}{k}$ as a **real polynomial** $f : \binom{[n]}{k} \to \{0, 1\}$ in n variables x_i.

If $i \in F$, then $x_i(F) = 1$. Otherwise, $x_i(F) = 0$.

Of course f is not unique. Choose f with $\deg(f)$ minimal.

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All k-sets **containing** 1: $f = x_1$.

All k-sets **containing 2 of** $\{1, 2, 3\}$: $f = x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3$.

A t**-junta** is a polynomial of **degree** t.

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Other structures:

- For vector spaces, take x_p as variables where p is a 1-spaces of $\mathbb{F}_q^n$.
  Here $x_p(F) = 1$ iff $p \subseteq F$.
- For permutations, take $x_{i \mapsto j}$ as variables.
  Here $x_{i \mapsto j}(\sigma) = 1$ iff $i^\sigma = j$.

Polynomials and Eigenspaces

Remark: I hope that at this point Karen Meagher and Nathan Lindzey talked exhaustively about eigenspaces/group characters.

Saying "f has **degree d**" same as saying "f in **certain eigenspaces**".

Sets: The Johnson graph on $\binom{[n]}{k}$ has $k+1$ eigenspaces V_i.

$$\underbrace{\overbrace{\underbrace{V_0}_{\text{deg}=0} + V_1}^{\text{deg}=1} + V_2}_{\text{deg}=2} + V_3 + \ldots + V_k$$

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Vector Spaces: Exactly the same for k-spaces as for k-sets.

Symmetric Group: Slightly different.

$$\underbrace{\overbrace{\underbrace{V_{[n]}}_{\text{deg}=0} + V_{[n-1,1]}}^{\text{deg}=1} + V_{[n-2,2]} + V_{[n-2,1,1]}}_{\text{deg}=2} + \ldots + V_{[1,1,\ldots,1]}$$

The Erdős-Ko-Rado Theorem

Theorem (Erdős-Ko-Rado (1961))

An intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ has at most size $\binom{n-1}{k-1}$ (for $n \geq 2k$).

Hoffman's **ratio bound** implies: Equality only if $\deg(f) \leq 1$ (for $n > 2k$).

Maybe more familiar: Equality iff the characteristic vector of $\mathcal{F}$ lies in $V_0 + V_1$.

Unique non-empty intersecting family $\mathcal{F}$ with $\deg(f) = 1$: $f = x_i$.

All k-sets which contain i.

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Vector spaces:

- All k-spaces through fixed 1-space p is largest.
- Equality if $\deg(f) \leq 1$, so if its characteristic vector is in $V_0 + V_1$.

Symmetric group:

- All permutations which map i to j is largest.
- Equality if $\deg(f) \leq 1$, so if its characteristic vector is in $V_{[n]} + V_{[n-1,1]}$.

The Hilton-Milner Theorem

Theorem (Hilton-Milner (1967))

Second largest maximal intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ has at most size $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ (for $n > 2k$).

Equality: $f = x_1 +$ terms of degree $\geq k-1$.

Here $F \in \mathcal{F}$ if (1) $1 \in F$ and $|F \cap \{2, \ldots, k+1\}| \geq 1$, or (2) $F = \{2, \ldots, k+1\}$.

Observation: Second largest family still almost degree 1.

In terms of eigenspaces, the characteristic vector of $\mathcal{F}$ lies "mostly" in $V_0 + V_1$.

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Similar results for **other structures**:

- Blokhuis, Brouwer, Chowdhury, Frankl, Mussche, Patkós, Szőnyi (2009), Blokhuis, Brouwer, Szőnyi (2012), Ih. (2019): for vector spaces.*
- Ellis (2012): for permutations.*

*: For both vector spaces as well as permutations, some cases are excluded.

All Large Intersecting Families

There are too many results for intersecting families. The following was observed by Dinur & Friedgut.

Theorem

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be intersecting. Then there exists a star $\mathcal{F}^*$ such that $|\mathcal{F} \setminus \mathcal{F}^*| \leq D\binom{n-2}{k-2}$.
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(Vague) **Idea:**

- An intersecting family of size $\geq C'\binom{n-2}{k-2}$ is almost degree 1.
  Much of $\mathcal{F}$ must be in $V_0 + V_1$.
- A degree 1 family is close to 1-junta.

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Ellis (2012): for **permutations**.

Vector spaces:

- Known for boring range of parameters.
- Unknown for interesting range of parameters.
  Let's say $n \in \{2k, 2k+1\}$.

Small Intersecting Families

Frankl, Kupavskii, others: Many degree versions of EKR for stability.
I believe that nothing on diversity has been done for $\mathrm{Sym}(n)$.

Theorem (Dinur, Friedgut (2009))

Fix $r \geq 2$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be intersecting of size $\geq c(r)\binom{n-r}{k-r}$.

Then there exists an $j(r)$-junta $\mathcal{F}^*$ such that $|\mathcal{F} \setminus \mathcal{F}^*| \leq c(r)\binom{n-r}{k-r}$.
Here k, n such that $1 < j(r) < k < n/2$.

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Here k, n such that $1 < j(r) < k < n/2$.

Idea same as before:

- An intersecting family of size $\geq c(r)\binom{n-r}{k-r}$ is close to degree $j'(r)$.
- An almost degree $j'(r)$-family is a $j(r)$-junta.

Vector spaces: investigated for easy cases only.

Permutations: follows from Ellis, Lifshitz (2019*).
Their result does much more.

Regular Intersecting Families

Juntas and **stars**: Some elements are **more important** than others.

Regular intersecting families of $\binom{[n]}{k}$:

- We want $|\{i \in F : F \in \mathcal{F}\}|$ be the same for all $i \in [n]$.
- This is a 1-design. That is $\mathcal{F}$ is orthogonal to V_1.

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Lovász (1977, in Hung., unpub.), Füredi (1981): $n \leq k^2 - k + 1$.

Equality occurs iff $\mathcal{F}$ is a **projective plane of order** $k-1$.

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Ih., Kupavskii (2019): some more results.

For instance, $|\mathcal{F}| \leq \frac{k-2}{2k-1}\binom{2k+1}{k}$ is a good bound for $n = 2k+1$.

Problem: No intersecting families in $V_1^{\perp}$ or $V_{[n-1,1]}^{\perp}$ in $\mathbb{F}_q^n$ or $\mathrm{Sym}(n)$!

Proof: MacWilliams transform/orthogonal projection onto V_1.

For $V_{[n-1,1]}$, I only checked small n.

Symmetric Intersecting Families

Symmetric intersecting families of $\binom{[n]}{k}$:

- We want $\mathrm{Aut}(\mathcal{F})$ to be transitive on $[n]$.
- This is a special case of a regular intersecting family.

Ellis, Kalai, Narayanan (2020): detailed investigation.

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This is also well-investigated for the **hypercube** $\mathcal{F} \subseteq [2]^n$!

Eberhard, Kahn, Narayanan, Spirkl (2019*): investigation for $\mathcal{F} \subseteq [q]^n$.

That is the q-Hamming graph.

Best known construction from **projective plane on n points**. (Compare to last slide.)

The Clique-Coclique Bound

Recall from **Karen's talk**:

I wrote this long before her talk. Hopefully, she discussed it.

Theorem (Clique-Coclique Bound (very specific case))

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be intersecting and $\mathcal{G} \subseteq \binom{[n]}{k}$ a partition of $[n]$ (here $k \mid n$). Then $|\mathcal{F}| \cdot \frac{n}{k} = |\mathcal{F}| \cdot |\mathcal{G}| \leq \binom{n}{k}$.

Proof: Use $|\mathcal{F} \cap \mathcal{G}^\sigma| \leq 1$ for $\sigma \in \mathrm{Sym}(n)$.

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Question: What if $\mathcal{F}$ has at most s pairwise disjoint sets?

So a matching has at most size s. Of course, $s = 1$ is intersecting.

Theorem (Clique-Coclique Bound (very specific case))

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ with matching number s (for $k \mid n$). Then $|\mathcal{F}| \leq s\binom{n-1}{k-1}$.

Proof: Use $|\mathcal{F} \cap \mathcal{G}^\sigma| \leq s$ for $\sigma \in \mathrm{Sym}(n)$.

Examples

Family 1: All k-sets which intersect $[s]$ non-trivially.

Family 2: All k-sets in $[(s+1)k-1]$.

Erdős conjectured (1965) that these are the largest:

It is not impossible that

$$f(n;r,k) = 1 + \max \left\{ \binom{rk-1}{r},\ g(n;r,k-1) \right\}. \tag{9}$$

For $r = 2$ (9) is implied by (1) and for $k=2$ (9) is proved in [2], but the general case seems elusive.

Note: The first family has almost degree 1 if $n \gg ks$.
The function $x_1 + x_2 + \ldots + x_s$ is a good approximation of it.

In the following, I **only care about** $n > (s+1)k$ (here Family 1 is larger).

History for $n > (s+1)k$

Erdős (1965): $n \geq n_0(k, s)$.

Bollobás, Daykin, Erdős (1976): $n \geq 2k^3 s$.

Frankl, Füredi (1990s?, unpub.): $n \geq cks^2$.

Huang, Loh, Sudakov (2012): $n \geq 3k^2 s$.

Frankl (2013): $n \geq (2s+1)k$.

Frankl, Kupavskii (2018*): $n \geq \frac{5}{3}sk$ for $s \geq s_0$.

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Ideas:

- Show that $\mathcal{F}$ contains a large intersecting family.
- Investigate intersection with partitions $\mathcal{G} \subseteq \binom{[n]}{k}$.

Ih. (2021): $16s \leq q^{\min(\frac{n-k}{4}, \frac{n-2k+1}{3})}$ for vector spaces.

My motivation: explore families with **degree close to 1** in $\mathbb{F}_q^n$.

Permutations

Let $\mathcal{F}$ be a family of permutation with matching number s.

Trivial Bound: $|\mathcal{F}| \leq s(n-1)!$.

Proof: $|\mathcal{F} \cap \mathcal{G}| \leq s$ for any derangement $\mathcal{G}$.

Equality: Take all permutations which map 1 to i for any $i \in [s]$.

There are **variants** to explore where above stops working:

- $\mathrm{PSL}(2, q)$ for $q \equiv 1 \pmod 4$. Here the bound stops working.
- G not t-transitive, t-intersecting-analog. Here the example stops working.

Back to Polynomials

Recall 1: We can write $\mathcal{F} \subseteq \binom{[n]}{k}$ as a real degree k polynomial.

Recall 2: This is the same as saying that the characteristic vector of $\mathcal{F}$ lies in certain eigenspaces.

$$\underbrace{\overbrace{\underbrace{V_0 +}_{\text{deg}=0} V_1}^{\text{deg}=1} + V_2}_{\text{deg}=2} + V_3 + \ldots + V_k$$

Recall 3: Previous objects have **low degree** or close to low degree.

Problem

What is the structure of low degree Boolean functions?

Hypercube

Well-investigated for the hypercube $\{0, 1\}^n$.

Trivial: A Boolean degree 1/affine function is one of $0, 1, x_i, 1 - x_i$.

Friedgut, Kalai, Naor (2002): A Boolean almost degree 1 function is close to one of $0, 1, x_i, 1 - x_i$.

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Friedgut, Kalai, Naor (2002): A Boolean almost degree 1 function is close to one of $0, 1, x_i, 1 - x_i$.

Nisan, Szegedi (1994): A Boolean degree d function is a $d2^{d-1}$-junta.

Chiarelli, Hatami, Saks (2018): Tight bound of $\Theta(2^d)$.

Kindler, Safra (2002, modified):
A Boolean almost degree d function is almost $O(2^d)$-junta.

Phrased with the Chiarelli, Hatamai, Saks (2018) bound.

Uniform Sets

Now look at $\mathcal{F} \subseteq \binom{[n]}{k}$.

Degree 1: A Boolean degree 1 function is one of $0, 1, x_i, 1 - x_i$.

Various proofs: Meyerowitz (1992, see Martin (2004)), Filmus (2016), De Boeck, Storme, Svob (2017), Filmus and Ih. (2019). The 1992, 2016, 2019 results show something more general. The 2017 result is only for $k \mid n$.

Filmus (2016):
A Boolean almost degree 1 function is almost a sum of x_i and $1 - x_i$.

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Filmus, Ih. (2019): A Boolean degree d function is a $O(2^d)$-junta.

Attention! Only if $\min(k, n - k)$ large enough!

Keller, Klein (2019):

A Boolean almost degree d function is a $O(2^d)$-junta.

Simplified. Similar restrictions apply.

Symmetric Group

Now $\mathcal{F} \subseteq \mathrm{Sym}(n)$.

Ellis, Friedgut, Pilpel (2011): A Boolean degree 1 function or its complement is a disjoint union of $x_{i \mapsto j}$.

Recall: $x_{i \mapsto j}(\sigma) = 1$ iff $i^\sigma = j$.

Ellis, Filmus, Friedgut (2015): A Boolean almost degree 1 function is close to a union of $x_{i \mapsto j}$.

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Dafni, Filmus, Lifshitz, Lindzey, Vinyals (2020*): More complex results for general degree d.

This is more complex and Nathan Lindzey might talk about this, so I leave it to him.

Problem: What about smaller simple groups?

Summary of all my suggested group problems in this talk: They are **solved for Sym(n)**, now do them for **other permutation groups**.

Vector Spaces

Now $\mathcal{F}$ a family of k-spaces of $\mathbb{F}_q^n$.

Recall: For a 1-space p, $x_p(S) = 1$ iff $p \subseteq S$.

Also write $x_H(S) = 1$ iff $S \subseteq H$ for a hyperplane H.

Famous among finite geometers: **Cameron-Liebler line classes.**

These are exactly **degree 1**.

Also special cases of: regular sets, equitable partititions, completely regular codes.

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Some **Degree 1:** x_p, x_H, $x_p + x_H$ for $p \not\subseteq H$ (and complements).
Read:

- All k-spaces containing p.
- All k-spaces in H.
- All k-spaces containing p or in H.

Conjecture (Cameron, Liebler (1982))

These are all nontrivial examples.

Technically, they made several conjectures, some true, and only for $k = 2$.

Cameron-Liebler Line Classes

Cameron-Liebler Classes of 2-spaces in $\mathbb{F}_q^4$: **well-investigated**.

Conjecture by Cameron and Liebler is **wrong here**.

Many counterexamples: Bruen, Cossidente, De Beule, Demeyer, Drudge, Feng, Gavrilyuk, Matkin, Metsch, Momihara, Pavese, Penttila, Rodgers, Xiang, Zou.

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Boolean degree 1 functions f on k-spaces for $n > 4$:

Theorem (Drudge (1998), Gavrilyuk and Mogilnykh (2014), Gavrilyuk and Matkin (2018), Matkin (2018))

All trivial *for* $k = 2$ *and* $q \leq 5$.

Theorem (Filmus, Ih. (2019))

All trivial *for* $k \geq 2$ *and* $q \leq 5$.

Also several existence conditions on the size of f by Blokhuis, De Boeck, D'haeseleer, Ih., Metsch, Rodgers, Storme, Vansweevelt (all recent).

Other Problems for Vector Space

What about almost degree 1, degree d, almost degree d?

All unclear!

Note: here "almost" is for q fixed and "some" asymptotics in n and maybe k.

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The **Unique Games Conjecture** claims that it is impossible to approximate many **NP-hard** problems in polynomial time.

Theorem (Khot, Minzer, Safra (2018))

The 2-to-2 Games Conjecture is true.

Proof: show that a family of k-spaces in $\mathbb{F}_2^n$ with **significant (Fourier) mass on low degree** is in some sense **far from regular**.

Open Problems

Recall: large intersecting families in $\binom{[n]}{k}$, $n > Ck$, are **close to juntas**.

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Characterize large intersecting families of k-spaces in $\mathbb{F}_q^{2k+1}$.

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For $\mathcal{F} \subseteq \mathrm{Sym}(n)$, define $\mathrm{a_{ij}} := |\{\sigma \in \mathcal{F} : \mathrm{i}^\sigma = \mathrm{j}\}|$. We call $\mathcal{F}$ **regular** if there ex. a, b and $\varphi \in \mathrm{Sym}(n)$ s.t. $a_{ij} = a$ if $i = j^\varphi$ and $a_{ij} = b$ otherwise.

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Investigate regular intersecting families in $\mathrm{Sym}(n)$.

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Find an interesting, feasible group case for the EM Conjecture.

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Problem

Study almost affine families of k-spaces in $\mathbb{F}_q^n$ *for* $n \to \infty$, k, q *fixed.*

Thank you for your attention!