On the intersection density of transitive groups

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Joint work with Karen Meagher and Pablo Spiga.

$ The Erdős-Ko-Rado theorem

(∗) A family $\mathcal{F}$ of $[n] := \{1, 2, \ldots, n\}$ is intersecting if for all $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

Theorem (Erdős-Ko-Rado(1961))

For any $n \geq 2k$, if $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. If $n \geq 2k+1$, then $|\mathcal{F}| = \binom{n-1}{k-1}$ if and only if $\mathcal{F}$ is $\mathcal{F}$ consists of all the k-subsets of $[n]$ containing a fixed element.

$ Intersecting permutations

(∗) A set $\mathcal{F} \subset \mathrm{Sym}(n)$ is intersecting if for any $\sigma, \pi \in \mathcal{F}$, there exist $i \in [n]$ such that $i^\sigma = i^\pi$.

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Theorem

Let $n \geq 2$.

* If $\mathcal{F} \subset \mathrm{Sym}(n)$ is intersecting, then $|\mathcal{F}| \leq (n-1)!$. (Deza-Frankl 1977)
* If $|\mathcal{F}| = (n-1)!$ if and only if $\mathcal{F}$ is a coset of a stabilizer of a point of $\mathrm{Sym}(n)$. (Cameron-Ku and Larose-Malvenuto 2004)

$$S_{i,j} = \{\sigma \in \mathrm{Sym}(n) \mid i^\sigma = j\}.$$

Can we extend the previous theorem for subgroups of $\mathrm{Sym}(n)$?

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No.

$ An example

* Consider the group $\mathrm{Alt}(4)$ with its natural action on $[4]$.
* Consider the induced action of $G = \mathrm{Alt}(4)$ on the six 2-subsets of $[4]$.
* Consider the isomorphic image $\widetilde{G}$ of G in $\mathrm{Sym}(6)$.
* A point stabilizer of $\widetilde{G}$ has size $\frac{|\widetilde{G}|}{6} = 2$.
* The family $\mathcal{F} = \{id, (1,2)(3,4), (3,4)(5,6), (1,2)(5,6)\} \subset \widetilde{G}$ is intersecting.

$ Definition

Let $G \leq \mathrm{Sym}(\Omega)$, $|\Omega| = n$, be a finite transitive group.

* The intersection density of an intersecting family $\mathcal{F} \subset G$ is the rational number

$$\rho(\mathcal{F}) := \frac{|\mathcal{F}|}{|G_\omega|},$$

  where $\omega \in \Omega$.

* The intersection density of a transitive group $G \leq \mathrm{Sym}(\Omega)$ is the rational number

$$\rho(G) := \max\{\rho(\mathcal{F}) : \mathcal{F} \subseteq G \text{ is intersecting}\}.$$

$ Proposition

Observation

Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group. Then, $\rho(G) \geq 1$ because G_ω is intersecting.

$ Definitions

Definition

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive.

* G has the EKR property if $\rho(G) = 1$,
* G has the strict EKR property if G has the EKR property and if $\mathcal{F}$ is an intersecting family such that $\rho(\mathcal{F}) = 1$, then $\mathcal{F}$ is a coset of a stabilizer of a point.

$ Some results in this area

Theorem (Meagher-Spiga-Tiep, 2015)

If G is a finite 2-transitive group, then G has the EKR property.

Theorem (Ellis-Friedgut-Pilpel, 2011)

Fix $t \in \mathbb{N}$. Then for n large enough depending on t, $\mathrm{Sym}(n)$ acting on the t-tuples of $[n]$ has the EKR property and the strict-EKR property.

Theorem (Ellis, 2012)

Fix $t \in \mathbb{N}$. Then for n large enough depending on t, $\mathrm{Sym}(n)$ acting on the t-subsets of $[n]$ has the EKR property and the strict-EKR property.

$ Open questions

Conjecture

For any $n \geq 2t+1$, $\mathrm{Sym}(n)$ acting on the t-tuples of $[n]$ has the EKR property and the strict-EKR property.

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For $t \leq n$, $\mathrm{Sym}(n)$ acting on the t-subsets of $[n]$ has the EKR property and the strict-EKR property (except for $(n,t) = (4,2),(5,2)$).

* $t=2$ K. Meagher and A. S. R
* $t=3$ A. Behajaina, R. Maleki, A. T. Rasoamanana, A. S. R.

$ Objective

Main goal

Classify the finite transitive groups that have the EKR property (strict-EKR property).

Derangement graphs

$ Derangement graph

If $G \leq \mathrm{Sym}(\Omega)$ is transitive and $\mathrm{Der}(G)$ is the set of all derangements of G, then the derangement graph Γ_G of G is the Cayley graph $\mathrm{Cay}(G, \mathrm{Der}(G))$. That is, Γ_G is the graph with

* vertex-set G,
* edge-set consisting of unordered pairs $(g,h) \in G \times G$ such that $hg^{-1} \in \mathrm{Der}(G)$.

$ Example of a derangement graph

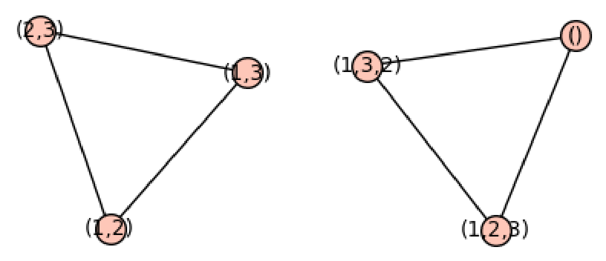


Figure: Derangement graph for Sym(3) with the natural action

$ Derangement graph for **Alt(4)**

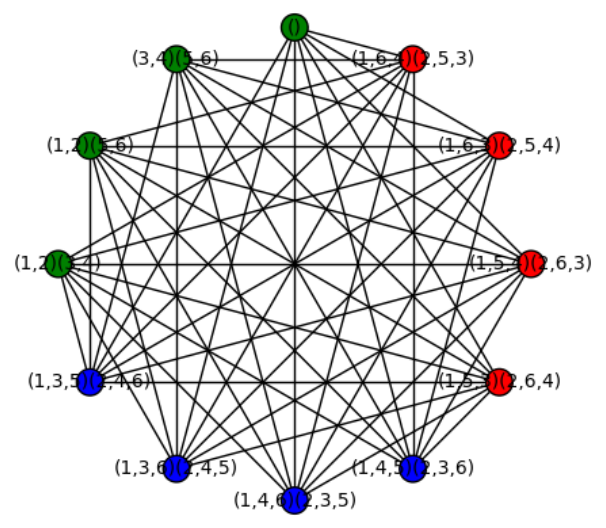


Figure: Derangement graph for Alt(4) acting on 6 points.

$ Derangement graph

$\mathcal{F} \subset G$ is intersecting

$\Leftrightarrow$

$\mathcal{F}$ is a coclique (independent set) of Γ_G.

Transitive groups that do not have the EKR property

$ Derangement graph for **Alt(4)**

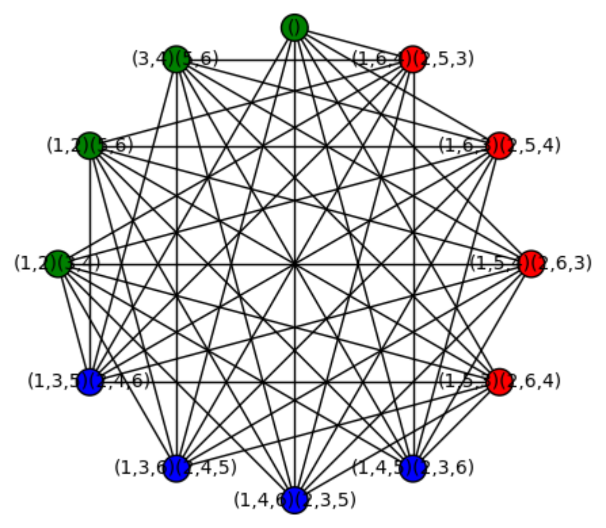
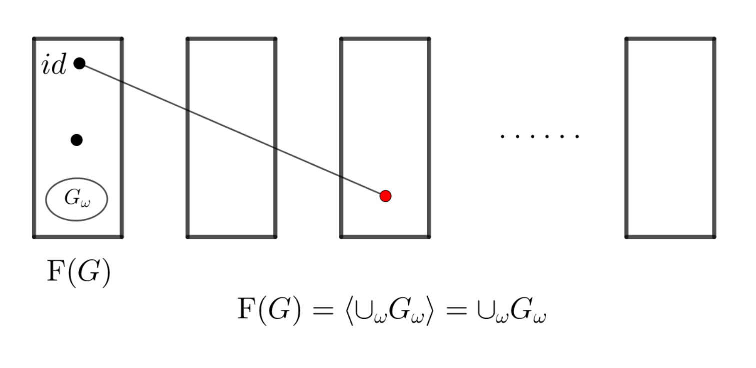


Figure: Derangement graph for Alt(4) acting on 6 points.

$ Complete multipartite derangement graphs



$ Remarks

In the library of TransitiveGroups of GAP and Sagemath, we found transitive groups whose derangement graphs were complete k-partite graph, for $k \in \{2,3,4,5,6,7,8,9,\dots\}$.

* There is only one transitive group with $k = 2$ (its degree is 2),
* there are only four groups with $k = 3$,
* No other transitive group with bipartite derangement graph.

$ Bipartite derangement graph

Theorem (Meagher, A.S.R., Spiga, 2021)

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive. Γ_G is bipartite if and only if $|\Omega| \leq 2$. Moreover, if $|\Omega| \geq 3$, then Γ_G has a triangle.

$ A direct consequence

Corollary (Meagher, A.S.R., Spiga, 2021)
If $G \leq \mathrm{Sym}(\Omega)$ is transitive, then $1 \leq \rho(G) \leq \frac{|\Omega|}{3}$.

$|\Omega| \geq 3$

$ A direct consequence

Corollary (Meagher,A.S.R., Spiga,2021)

If $G \leq \mathrm{Sym}(\Omega)$ is transitive, then $1 \leq \rho(G) \leq \frac{|\Omega|}{3}$.

Conjecture 1

$\rho(G) = \frac{|\Omega|}{3}$ if and only if Γ_G is a complete 3-partite graph.

$ Complete multipartite derangement graphs

Conjecture 2

For any $k \geq 3$, there exists a transitive group G of degree $n(k)$ such that Γ_G is complete k-partite.

Problem 1

Find more transitive groups whose derangement graphs are complete 3-partite graphs.

$ Structure of derangement graphs

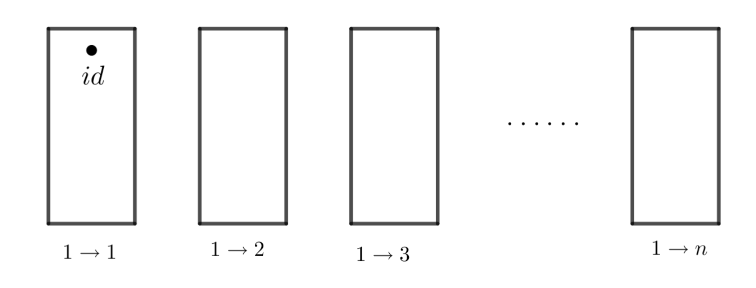
Question 1

Is it possible that Γ_G is a primitive strongly regular graph, for some finite transitive group G?

Chromatic number

$ A natural representation

Let G be a transitive group of degree n.



$ Chromatic number

Proposition

Let G be a transitive group of degree n

* $\chi(\Gamma_G) \leq n$.,
* If G has the EKR property, then $\chi(\Gamma_G) = n$.

Question 2

Does there exist a transitive group G of degree n such that $\chi(\Gamma_G) = n$ and G does not have the EKR property?

Intersection density of transitive groups

$ Intersection density of groups of a given degree

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive.

Lemma

If $|\Omega| = p$ is prime, then $\rho(G) = 1$.

Theorem (Li-Song-Pantangi, 2020)

If $|\Omega| = p^k$ is a prime power, then $\rho(G) = 1$.

Theorem (ASR, 2021)

If $|\Omega| = 2p$, where p is an odd prime, then $\rho(G) \in \mathbb{Q} \cap [1,2]$.

Theorem (Marusic et al., 2021)

If $|\Omega| = 2p$, where p is an odd prime, then $\rho(G) \in \{1,2\}$.

$ Conjectures

Conjecture 3

Let p and q be distinct odd primes. If $G \leq \mathrm{Sym}(\Omega)$ is transitive of degree pq, then $\rho(G) = 1$.

The affine groups

$ Definitions

For any $A \in \mathrm{GL}(2, q)$ and $b \in \mathbb{F}_q^2$, an affine transformation of $\mathbb{F}_q^2$

$$\begin{aligned}(b, A) : \mathbb{F}_q^2 &\to \mathbb{F}_q^2 \\ v &\mapsto Av + b.\end{aligned}$$

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$$\begin{aligned}(b,A) : \mathbb{F}_q^2 &\to \mathbb{F}_q^2 \\ v &\mapsto Av + b.\end{aligned}$$

The affine group $\mathrm{AGL}(2,q) := \left\{(b,A) \mid A \in \mathrm{GL}(2,q), b \in \mathbb{F}_q^2\right\}$ is the group with multiplication

$$(b_1, A_1)(b_2, A_2) = (A_1 b_2 + b_1, A_1 A_2).$$

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The group structure is

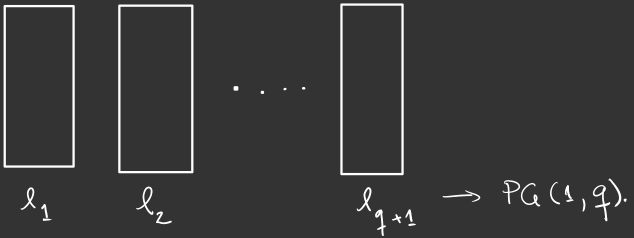
$$\mathrm{AGL}(2,q) = \mathbb{F}_q^2 \rtimes \mathrm{GL}(2,q).$$

$ Action of $\mathbf{AGL(2, q)}$

* $\mathrm{AGL}(2, q)$ acts naturally on $\mathbb{F}_q^2$ (the points of $\mathrm{AG}(2, q)$).
* This action is 2-transitive.

$ Action of **AGL$(2,q)$**

* AGL$(2,q)$ acts naturally on $\mathbb{F}_q^2$ (the points of AG$(2,q)$).
* This action is 2-transitive.
* AGL$(2,q)$ acts on the lines of AG$(2,q)$.
* The action on the lines is rank 3 imprimitive. The unique system of imprimitivity is induced by PG$(1,q)$.



$ Blocks and stabilizer of the blocks

* Let Δ be the unique system of imprimitivity of $\mathrm{AGL}(2,q)$.
* The kernel of the action of $\mathrm{AGL}(2,q)$ on Δ is

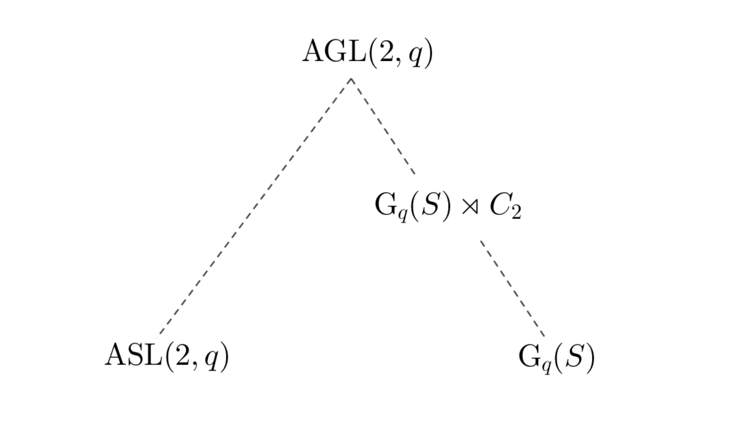
$$H_q \cong \mathbb{F}_q^2 \rtimes \mathbb{F}_q^*.I_2.$$

$$= \{(b, A) \in AGL(2,q) \mid b \in \mathbb{F}_q^2,\ A = \lambda I\} \quad (\lambda \in \mathbb{F}_q^*)$$

$ A special affine group

* Consider a Singer cycle S of $\mathrm{GL}(2,q)$.
* Let $G_q(S) := \left\{ (b, A) \mid b \in \mathbb{F}_q^2, A \in \langle S \rangle \right\}$.
* $\Gamma_{G_q(S)}$ is complete $(q+1)$-partite graph.
* $\rho(G_q(S)) = \frac{q^2(q-1)}{q(q-1)} = q$.

$ Lattice of subgroups



$ Intersection density of AGL(2, q)

* We have $\rho(\mathrm{AGL}(2,q)) \le \rho(G_q(S)) = q$.
* $\rho(\mathrm{AGL}(2,3)) = \frac{45}{36}$ and $\rho(\mathrm{AGL}(2,4)) = 192/144$.

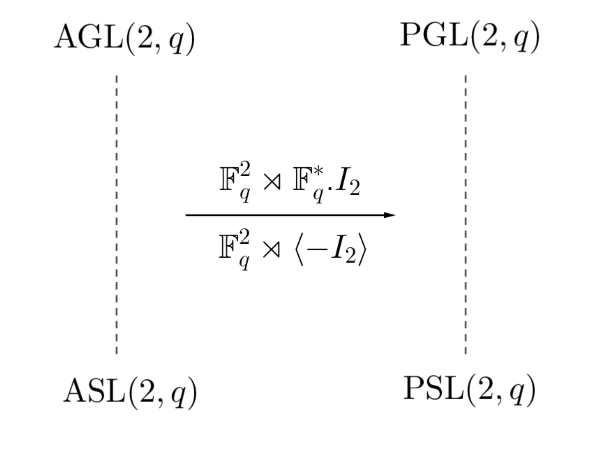
Theorem

$\rho(\mathrm{AGL}(2,q)) > 1$.

Conjecture 4

For any $\epsilon > 0$, $\exists q_0 > 0$ $(q \ge q_0) \Rightarrow \rho(\mathrm{AGL}(2,q)) - 1 < \varepsilon$.
Moreover, if $\mathcal{F} \subset \mathrm{AGL}(2,q)$ is intersecting then $|\mathcal{F}| \le f(q)|\mathbb{F}_q^2 \rtimes \mathbb{F}_q^*.I|$, where $f(q)$ is linear in q.

$ Connection to $\mathbf{PGL(2,q)}$



$ Connection to $\mathbf{PGL(2,q)}$

Lemma

If $\mathcal{F} = \{A_1, A_2, \ldots, A_\ell\} \subset \mathrm{PGL}(2,q)$ is 2-intersecting, then $\mathcal{Q} = (0, A_1)H_q \cup (0, A_2)H_q \cup \ldots \cup (0, A_\ell)H_q \subset \mathrm{AGL}(2,q)$ is intersecting.

Proof.

* There are no edges between $(0, A_i)H_q$ and $(0, A_j)H_q$ iff $(0, A_j^{-1}A_i)$ intersects with every $h = (b, kI_2) \in H_q$.
* By assumption, there exist $\ell, \ell' \in \mathrm{PG}(1,q)$ such that

$$\begin{cases} A_j^{-1}A_i\ell = \ell \\ A_j^{-1}A_i\ell' = \ell'. \end{cases}$$

* $(b, kI_2)^{-1}(0, A_j^{-1}A_i) = (-\frac{1}{k}b, \frac{1}{k}A_j^{-1}A_i)$ fixes a line.

$ Open question

Problem

Find a sharp upper bound on the size of the maximum cocliques of $\mathrm{AGL}(2,q)$.

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Thank you!!