Instantaneous Uniform Mixing

Open Problems in Algebraic Graph Theory

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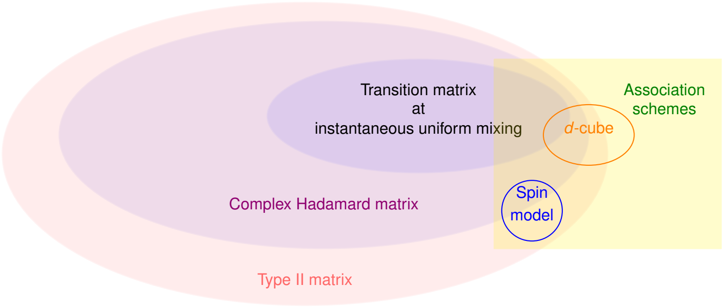

Transition matrix
at
instantaneous uniform mixing
Association
schemes
d-cube
Spin
model
Complex Hadamard matrix
Type II matrix

Definition

Let X be a graph with adjacency matrix A.

The continuous-time quantum walk on X is given by the transition operator $U(t) := e^{-itA}$.

Given the spectral decomposition of A:

$$A = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d,$$

we have

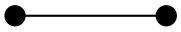
$$U(t) = e^{-it\theta_0} E_0 + e^{-it\theta_1} E_1 + \cdots + e^{-it\theta_d} E_d.$$

Definition

Instantaneous uniform mixing occurs at time τ if

$$|U(\tau)_{u,w}| = \frac{1}{\sqrt{|V(X)|}}, \qquad \text{for all } w, u \in V(X).$$

Example: K_2



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (1) \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + (-1) \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\implies \quad U(t) = e^{-it} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + e^{it} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos t & -i \sin t \\ -i \sin t & \cos t \end{bmatrix}.$$

At time $\frac{\pi}{4}$: $\quad U(\frac{\pi}{4}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ instantaneous uniform mixing

Definition

Perfect state transfer occurs between a and b at time τ if $U(\tau)e_a = \beta e_b$, that is,

$$U(\tau) = \begin{bmatrix} 0 & \beta & 0 & \cdots & 0 \\ \beta & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix}, \qquad \text{for some } |\beta| = 1.$$

K_2 at time $\frac{\pi}{2}$: $\quad U(\frac{\pi}{2}) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ perfect state transfer

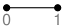
Definition

Let X and Y be graphs with m and n vertices, respectively.

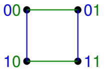
The Cartesian product of X and Y, denoted by $X \,\square\, Y$ is the graph with

- vertex set: $V(X) \times V(Y)$
- edges: $(u_1, v_1) \sim (u_2, v_2)$ if $u_1 \sim u_2$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1 \sim v_2$,
- adjacency matrix: $A_{X \,\square\, Y} = A_X \otimes \mathrm{I}_n + \mathrm{I}_m \otimes A_Y$.

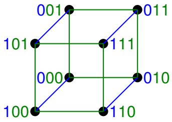
1-cube



1-cube $\square$ 1-cube



1-cube $\square$ 2-cube



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Observe: d-cube $\cong$ 1-cube$^{\square\, d}$, for $d \geq 2$.

Transition matrix of $X \,\square\, Y$:

$$\begin{aligned} U_{X \,\square\, Y}(t) &= e^{-it\,(A_X \otimes \mathrm{I} + \mathrm{I} \otimes A_Y)} \\ &= e^{-it\,(A_X \otimes \mathrm{I})} \cdot e^{-it\,(\mathrm{I} \otimes A_Y)} \\ &= \left(e^{-itA_X} \otimes \mathrm{I}\right)\left(\mathrm{I} \otimes e^{-itA_Y}\right) \\ &= e^{-itA_X} \otimes e^{-itA_Y} \\ &= U_X(t) \otimes U_Y(t) \end{aligned}$$

For d-cubes: $U(t) = \begin{bmatrix} \cos t & -i \sin t \\ -i \sin t & \cos t \end{bmatrix}^{\otimes d}$

- Instantaneous uniform mixing at time $\frac{\pi}{4}$: $U(\frac{\pi}{4}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\otimes d}$
- Perfect state transfer at time $\frac{\pi}{2}$: $U(\frac{\pi}{2}) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}^{\otimes d}$

Graphs with instantaneous uniform mixing:

- non-regular graph: $K_{1,3}$;
- complete graphs: K_2, K_3 and K_4;
- Hamming graphs: $K_2^{\square d}$ (d-cube), $K_3^{\square d}$ and $K_4^{\square d}$;
- Paley graphs of order 9;
- strongly regular graphs where $J - 2A$ is a regular symmetric Hadamard matrix of order $4\theta^2$, for even θ.
- strongly regular graphs where $J - 2A - 2I$ is a regular symmetric Hadamard matrix of order $4\theta^2$, for odd θ.
- some integral Cayley graphs over $\mathbb{Z}_2^d$, $\mathbb{Z}_3^d$, $\mathbb{Z}_4^d$ or $\mathbb{Z}_2^r \otimes \mathbb{Z}_4^s$; and
- the Cartesian products of graphs admitting instantaneous uniform mixing at the same time.

Problem: Is there any non-regular graph, other than $K_{1,3}$ or its Cartesian products, that have instantaneous uniform mixing?

Graphs that do not admit instantaneous uniform mixing:

- Integral Cayley graphs on Γ when $\Gamma \neq \mathbb{Z}_2^d, \mathbb{Z}_3^d, \mathbb{Z}_4^d, \mathbb{Z}_2^r \otimes \mathbb{Z}_4^s$; (Xiwang Cao)

  Problem: Can instantaneous uniform mixing occur in non-integral Cayley graphs on abelian groups?

- Cycle C_n when $n > 4$ is even or n is an odd prime; (Natalie Mullin)

  Problem: Can instantaneous uniform mixing occur in odd cycles? C_9?

- Normal Cayley graphs of an extraspecial p-group. (Peter Sin)

  Problem: Can instantaneous uniform mixing occur in Cayley graphs on non-abelian groups?

(Christino Tamon - applications of mixing)

Recall: Instantaneous uniform mixing occurs at time τ if

$$|U(\tau)_{u,w}| = \frac{1}{\sqrt{|V(X)|}}, \qquad \text{for all } w, u \in V(X).$$

K_2 at time $\frac{\pi}{4}$: $\quad U(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

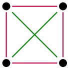
Let $H = \sqrt{v}U(\tau)$. Then

- $|H_{w,u}| = 1$.
- $U(t)$ is unitary $\implies$ $\overline{H}^T H = v\, \overline{U(\tau)}^T U(\tau) = v\, \mathrm{I}$.
- H is called a complex Hadamard matrix.

Theorem

If instantaneous uniform mixing occurs in X then the adjacency algebra $\langle A \rangle$ contains a complex Hadamard matrix.


Association
schemes
Complex Hadamard matrix

Example: C_4: 

Let $\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}$

Observe:

- $A_0 = I$.
- $A_0 + A_1 + A_2 = J$.
- $A_j \in \mathcal{A} \implies A_j^T \in \mathcal{A}, \quad \forall j$.
- $\begin{cases} A_1^2 = 2A_0 + 2A_2 \\ A_1 A_2 = A_2 A_1 = A_1 \\ A_2^2 = A_0 \end{cases} \implies A_j A_k \in \operatorname{span} \mathcal{A}, \quad \forall j, k$.
- $A_j A_k = A_k A_j, \quad \forall j, k$.

Definition

An association scheme with d classes is a set of $v \times v$ $(0,1)$-matrices $\{A_0, A_1, \ldots, A_d\}$ satisfying

- $A_0 = I$.
- $\sum_j A_j = J$ (span$\mathcal{A}$ is closed under the Schur (entrywise) product $\circ$)
- $A_j \in \mathcal{A} \Longrightarrow A_j^T \in \mathcal{A}, \quad \forall j$ (span$\mathcal{A}$ is closed under taking transpose)
- $A_j A_k \in \text{span } \mathcal{A}, \quad \forall j, k$ (span$\mathcal{A}$ is closed under matrix multiplication)
- $A_j A_k = A_k A_j, \quad \forall j, k$ (span$\mathcal{A}$ is commutative)

The span of $\mathcal{A}$ is called the Bose-Mesner algebra of $\mathcal{A}$.

Example: Distance regular graphs

- Complete graph: $\mathcal{A} = \{\mathrm{I}, J - \mathrm{I}\}$ (Tamon et al.)
- d-cube: A_j is the j-th distance matrix, for $j = 1, \ldots, d$. (Russell and Moore)
- Cycle C_n: A_j is the j-th distance matrix, for $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$. (Mullin)

Two bases of $\mathrm{span}\mathcal{A}$:

- Schur idempotents: $\{A_0, A_1, \ldots, A_d\}$
  - $\displaystyle\sum_{r=0}^{d} A_r = J$
  - $A_j \circ A_k = \delta_{j,k} A_j$
  - $A_j A_k = A_k A_j$
- Principal idempotents: $\{E_0, E_1, \ldots, E_d\}$
  - E_s is the projection matrix of the s-th common eigenspace of the A_r's
  - $\displaystyle\sum_{r=0}^{d} E_r = \mathrm{I}$
  - $E_j \cdot E_k = \delta_{j,k} E_j$
- Eigenvalues of $\mathcal{A}$: $A_r E_s = p_r(s) E_s$, for $r, s = 1, \ldots, d$.

Matrix of eigenvalues:

$$\begin{array}{c} \\ E_0 \\ E_1 \\ \vdots \\ E_d \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{I} & A_1 & & A_d \end{array} \\ \begin{bmatrix} 1 & p_1(0) & \ldots & p_d(0) \\ 1 & p_1(1) & \ldots & p_d(1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_1(d) & \ldots & p_d(d) \end{bmatrix} \end{array}$$

$$A_r = \sum_{s=0}^{d} p_r(s) E_s$$

Search for complex Hadamard matrices in $\mathcal{A}$: $$W = \sum_{j=0}^{d} t_j\, A_j = \sum_{k=0}^{d} \left(\sum_{j=0}^{d} p_j(k) t_j \right) E_k$$

Then W is a complex Hadamard matrix

$$\iff \quad \overline{W}^T\, W = v\,\mathrm{I}$$

$$\iff \quad \left[\sum_{s=0}^{d} \left(\sum_{r=0}^{d} p_r(s) \overline{t_r} \right) E_s \right] \cdot \left[\sum_{j=0}^{d} \left(\sum_{k=0}^{d} p_j(k) t_j \right) E_k \right] = v\,\mathrm{I}$$

$$\iff \quad \sum_{s=0}^{d} \left(\sum_{r=0}^{d} p_r(s) \overline{t_r} \right) \left(\sum_{j=0}^{d} p_j(s) t_j \right) E_s = v \sum_{s=0}^{d} E_s$$

$$\iff \quad \left(\sum_{r=0}^{d} p_r(s) \overline{t_r} \right) \left(\sum_{j=0}^{d} p_j(s) t_j \right) = v, \qquad \forall s.$$

For C_4: $$\begin{cases} \left(\overline{t_0} + 2\overline{t_1} + \overline{t_2}\right) (t_0 + 2t_1 + t_2) = 4 \\ \left(\overline{t_0} - \overline{t_2}\right) (t_0 - t_2) = 4 \\ \left(\overline{t_0} - 2\overline{t_1} + \overline{t_2}\right) (t_0 - 2t_1 + t_2) = 4 \end{cases}$$

An inequality:

$$
\begin{aligned}
v &= \left(\sum_{r=0}^{d} p_r(s)\overline{t_r}\right)\left(\sum_{j=0}^{d} p_j(s)t_j\right) \\
&= \left|\sum_{r=0}^{d} p_r(s)^2 + \sum_{r<j}\left(\overline{t_r}t_j + \overline{t_j}t_r\right)p_r(s)p_j(s)\right| \\
&\leq \sum_{r=0}^{d} |p_r(s)|^2 + \sum_{r<j} 2\,|p_r(s)|\cdot|p_j(s)| \\
&= \left(\sum_{r=0}^{d} |p_r(s)|\right)^2 \qquad \text{for } s = 0, \ldots, d.
\end{aligned}
$$

For K_v:

$$
\begin{matrix} s=0 \\ s=1 \end{matrix}\quad \begin{bmatrix} 1 & v-1 \\ 1 & -1 \end{bmatrix} \quad \Longrightarrow \quad v \leq 4
$$

Theorem (Tamon et al. 2008)

The Hamming graph $K_v^{\square\, d}$ admits instantaneous uniform mixing if and only if $v \leq 4$.

Strongly regular graphs:

Theorem

The Bose-Mesner algebra of a strongly regular graph contains a complex Hadamard matrix if and only if it has one of the following parameters:

a. $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$

b. $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$

c. $(4\theta^2 - 1, 2\theta^2, \theta^2, \theta^2)$

d. $(4\theta^2 + 4\theta + 1, 2\theta^2 + 2\theta, \theta^2 + \theta - 1, \theta^2 + \theta)$

e. $(4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2)$

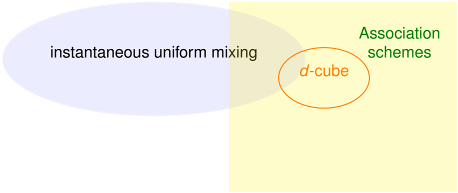
Theorem (Godsil, Mullin, Roy 2017)

A primitive strongly regular graph admits instantaneous uniform mixing if and only if one of the following holds

a. *it has parameter set* $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$*, for even* θ*;*

b. *it has parameter set* $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$*, for odd* θ*;*

c. *it is the Paley graph of order* 9 *which has parameters* $(9, 4, 1, 2)$.


instantaneous uniform mixing
Association schemes
d-cube

d-cubes: $\mathcal{A} = \{\mathrm{I}, A_1, \ldots, A_d\}$

- Eigenvalues: $p_r(s) = \sum_{h=0}^{r} (-2)^h \binom{d-h}{r-h} \binom{s}{h}$ (Krawtchouk polynomials $p_r(x)$)

- For A_1: $U(\frac{\pi}{4}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\otimes d}$

- $\begin{bmatrix} \frac{1}{\sqrt{2}} & \pm\frac{i}{\sqrt{2}} \\ \pm\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\otimes d} \in \mathrm{span}\mathcal{A}$

Question: Can other graphs in $\mathrm{span}\mathcal{A}$ admit instantaneous uniform mixing?

Idea: Find a 01-matrix $M \in \mathcal{A}$ satisfying $e^{-i\tau M} = \frac{e^{i\beta}}{\sqrt{2}^d} \begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}^{\otimes d}$, for some time τ and $\beta \in \mathbb{R}$. $(*)$

Eigenvalues: Suppose $M = \sum_{s=0}^{d} \theta_s E_s$.

Then $e^{-i\tau M} E_s = e^{-i\tau\theta_s} E_s$ and

$$\frac{e^{i\beta}}{\sqrt{2}^d} \begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}^{\otimes d} E_s = e^{i\beta} e^{\pm i\pi\left(\frac{d-2s}{4}\right)} E_s, \qquad \forall s.$$

$$\begin{aligned} (*) \text{ holds} \iff \quad e^{-i\tau\theta_s} &= e^{i\beta} e^{\pm i\pi\left(\frac{d-2s}{4}\right)}, \qquad \forall s \\ \iff \quad -\tau\theta_s &= \pm\pi\frac{(d-2s)}{4} + \beta \pmod{2\pi}, \qquad \forall s \\ \iff \quad \tau(\theta_s - \theta_{s-1}) &= \pm\frac{\pi}{2} \pmod{2\pi}, \qquad \forall s > 0 \end{aligned}$$

Example: 4-cube and pick time $\tau = \frac{\pi}{4}$

$$\frac{\pi}{4}(\theta_s - \theta_{s-1}) = \pm\frac{\pi}{2} \pmod{2\pi} \iff (\theta_s - \theta_{s-1}) = \pm 2 \pmod 8$$

	I	A_1	A_2	A_3	A_4
θ_0	1	4	6	4	1
θ_1	1	2	0	-2	-1
θ_2	1	0	-2	0	1
θ_3	1	-2	0	2	-1
θ_4	1	-4	6	-4	1

Both G_1 and G_3 admit instantaneous uniform mixing at $\frac{\pi}{4}$.

Theorem

For even d and odd r, G_r admits instantaneous uniform mixing at time $\frac{\pi}{4}$.

Faster $\tau = \frac{\pi}{8}$: $\frac{\pi}{2^3}(\theta_s - \theta_{s-1}) = \pm\frac{\pi}{2} \pmod{2\pi} \iff (\theta_s - \theta_{s-1}) = \pm 2^2 \pmod{2^4}$

24-cube: $\theta_s - \theta_{s-1} \pmod{16}$

A_i

E_s

1	8	4	8	2	8	4	8	15	0	8	0	12	0	8	**0**	15	8	4	8	2	8	4	8
1	6	6	14	12	10	2	2	5	12	12	12	0	4	4	**4**	11	14	14	6	4	2	10	10
1	4	12	12	2	12	12	4	15	8	8	8	12	8	8	**8**	15	4	12	12	2	12	12	4
1	2	6	10	12	14	2	6	5	4	12	4	0	12	4	**12**	11	10	14	2	4	6	10	14
1	0	4	0	2	0	4	0	15	0	8	0	12	0	8	**0**	15	0	4	0	2	0	4	0
1	14	6	6	12	2	2	10	5	12	12	12	0	4	4	**4**	11	6	14	14	4	10	10	2
1	12	12	4	2	4	12	12	15	8	8	8	12	8	8	**8**	15	12	12	4	2	4	12	12
1	10	6	2	12	6	2	14	5	4	12	4	0	12	4	**12**	11	2	14	10	4	14	10	6
1	8	4	8	2	8	4	8	15	0	8	0	12	0	8	**0**	15	8	4	8	2	8	4	8
1	6	6	14	12	10	2	2	5	12	12	12	0	4	4	**4**	11	14	14	6	4	2	10	10
1	4	12	12	2	12	12	4	15	8	8	8	12	8	8	**8**	15	4	12	12	2	12	12	4
1	2	6	10	12	14	2	6	5	4	12	4	0	12	4	**12**	11	10	14	2	4	6	10	14
1	0	4	0	2	0	4	0	15	0	8	0	12	0	8	**0**	15	0	4	0	2	0	4	0
1	14	6	6	12	2	2	10	5	12	12	12	0	4	4	**4**	11	6	14	14	4	10	10	2
1	12	12	4	2	4	12	12	15	8	8	8	12	8	8	**8**	15	12	12	4	2	4	12	12
1	10	6	2	12	6	2	14	5	4	12	4	0	12	4	**12**	11	2	14	10	4	14	10	6
1	8	4	8	2	8	4	8	15	0	8	0	12	0	8	**0**	15	8	4	8	2	8	4	8
1	6	6	14	12	10	2	2	5	12	12	12	0	4	4	**4**	11	14	14	6	4	2	10	10
1	4	12	12	2	12	12	4	15	8	8	8	12	8	8	**8**	15	4	12	12	2	12	12	4
1	2	6	10	12	14	2	6	5	4	12	4	0	12	4	**12**	11	10	14	2	4	6	10	14
1	0	4	0	2	0	4	0	15	0	8	0	12	0	8	**0**	15	0	4	0	2	0	4	0
1	14	6	6	12	2	2	10	5	12	12	12	0	4	4	**4**	11	6	14	14	4	10	10	2

Nutrition Facts/Valeur nutritive	
per *n*-cube	
Calories/Calories	$\pm$?
$\vdots$	$\vdots$
Carbohydrate/Glucides	> 5g
Fibre/Fibres	0g
Sugars	> 5g
$\vdots$	$\vdots$

INGREDIENTS: continuous-time quantum walk, $p_r(s)$ (Krawtchouk polynomials), spectral decomposition, *number theory (Lucas' Theorem, Kummer's Theorem)*, goos paper, snacks, teabags.

Theorem

For $k \geq 2$, the $(2^{k+1} - 1)$-distance graph of the $(2^{k+2} - 8)$-cube admits instantaneous uniform mixing at $\frac{\pi}{2^k}$.

Double the time:

$$\left(\frac{1}{\sqrt{2}^d}\begin{bmatrix}1 & \pm i \\ \pm i & 1\end{bmatrix}^{\otimes d}\right)^2 = \begin{bmatrix}0 & \pm i \\ \pm i & 0\end{bmatrix}^{\otimes d}$$

$$U(\tau) = \frac{1}{\sqrt{2}^d}\begin{bmatrix}1 & \pm i \\ \pm i & 1\end{bmatrix}^{\otimes d}$$

Instantaneous uniform mixing

$$\Longrightarrow \quad U(2\tau) = \begin{bmatrix}0 & \pm i \\ \pm i & 0\end{bmatrix}^{\otimes d}$$

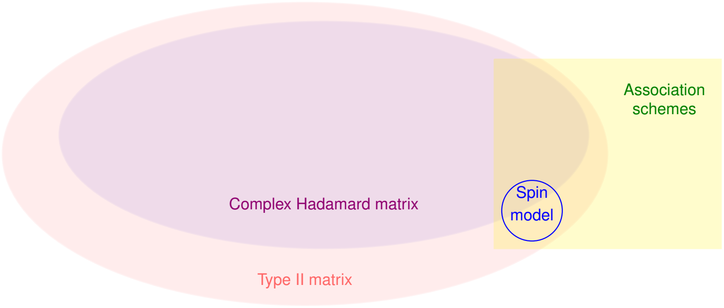
perfect state transfer

Theorem

For $k \geq 2$, the $(2^{k+1}-1)$-distance graph of the $(2^{k+2}-8)$-cube admits perfect state transfer at $\frac{\pi}{2^{k-1}}$.

Problems:

- Does a cubelike graph has perfect state transfer at time 2τ if instantaneous uniform mixing occurs at time τ?
- Characterize the connection set of a cubelike graph having instantaneous uniform mixing.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Complex Hadamard matrix
Type II matrix

The catalogue: https://chaos.if.uj.edu.pl/ karol/hadamard/

catalogue

- news
- definitions
- theory
- scripts
- game
- literature
- links
- acknowledgements
- v6.2021.01.14 • contact

Hadamard 2022

Complex Hadamard Matrices

a catalogue (since 2006)

by Wojciech Bruzda, Wojciech Tadej and Karol Życzkowski

old version

including Butson-type matrices

Butson type matrices are listed in a dephased, log-Hadamard form.

For any representant of an affine Hadamard family $H(a) = F \circ \mathrm{EXP}(ia)$ being a Butson type $H(a) \in BH(N, q)$ there exists another parameter $b = 2\pi/kq : k \in \mathbb{Z}$ such that $H(b) \in BH(N, qk)$. Due to this fact the set of all Butsons is countable but not finite. Thus from an affine family of complex Hadamard matrices we provide only the Butson representant with the minimal factor k.

Comprehensive study of Butson type matrices of size $N \leqslant 21$ can be found in the `Butson Home` by P. Lampio and F. Szöllősi.

$F_2^{(0)} = H_2 \supset F_2 \in BH(2,2)$	generic defect values = $\{0\}$
$F_3^{(0)} = F_3 \in BH(3,3)$	$\{0\}$
$H_4 \simeq F_2 \otimes F_2 \in BH(4,2)$	$\{3\}$
$F_4^{(1)} \supset F_4 \in BH(4,4)$	$\{1\}$

Connections to von-Neumann algebras, error correcting codes, equiangular lines and mutually unbiased bases ......

Problem: Find complex Hadamard matrices in association schemes. (Ikuta and Munemasa; Kharaghani et al.)

Definition

A $v \times v$ matrix W is type II if

$$W^{-1} = \frac{1}{v} \begin{pmatrix} \frac{1}{W_{1,1}} & \frac{1}{W_{1,2}} & \cdots & \frac{1}{W_{1,v}} \\ \frac{1}{W_{2,1}} & \frac{1}{W_{2,2}} & \cdots & \frac{1}{W_{2,v}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{W_{v,1}} & \frac{1}{W_{v,2}} & \cdots & \frac{1}{W_{v,v}} \end{pmatrix}^T$$

Example:

- Complex Hadamard matrices.
  - $U(\tau)$ at instantaneous uniform mixing
  - Hadamard matrices: $H_{u,w} = \pm 1$ and $H^T H = v\,\mathrm{I}$
  - Character tables of finite abelian groups.
- Potts model: $W = -t^3\mathrm{I} + \frac{1}{t}(J - \mathrm{I}), \quad 2 - t^4 + t^{-4} = v.$ (link invariants)
- Two distinct entries $\iff$ Hadamard matrices or symmetric designs.
- Tight set of equiangular lines $\implies$ type II matrices with quadratic minimal polynomials.

Definition

Let W be a type II matrix.
Define

$$Y_{ab} = \begin{bmatrix} \frac{W_{1,a}}{W_{1,b}} \\ \frac{W_{2,a}}{W_{2,b}} \\ \vdots \\ \frac{W_{v,a}}{W_{v,b}} \end{bmatrix}, \qquad \forall a, b.$$

The Nomura algebra of W is $\mathcal{N}_W = \{M \ : \ Y_{ab} \text{ is an eigenvector of M}, \forall a, b\}$

Theorem (Jaeger, Matsumoto, Nomura 1998)

Let W be a type II matrix.

Then $\mathcal{N}_W$ and $\mathcal{N}_{W^T}$ are the Bose-Mesner algebras of a formally dual pair of association schemes.

Example:

- Potts model: $\mathcal{N}_W = \text{span}\{I, J\}$
- Symmetric designs: $\mathcal{N}_W = \text{span}\{I, J\}$
- Hadamard matrices:
  - $v = 4 \pmod 8$: $\mathcal{N}_W = \text{span}\{I, J\}$
  - $v = 0 \pmod 8$: Open

Problem: Find type II matrices with non-trivial Nomura algebras.

Theorem (Jaeger, Matsumoto, Nomura 1998)

Let W be a type II matrix.
Then $W \in \mathcal{N}_W$ if and only if cW is a spin model, for some $c \in \mathbb{C}$.

- 1-class schemes $\{I, J - I\}$: Potts model — Jones polynomial
- symmetric 2-class schemes: Jaeger (1992) — Kauffman polynomial
- symmetric 3-class schemes: Open

Theorem (Godsil and Roy 2009)

Given a unitary *spin model W, let D_j be the diagonal matrix whose (r, r)-entry is $\sqrt{v}W_{r,j}$.*
Then $\{I, W, D_jW\}$ form a set of three mutually unbiased bases.

Problems:

- Is there any non-regular graph, other than $K_{1,3}$ or its Cartesian products, that have instantaneous uniform mixing?
- Is C_3 the only odd cycle admitting instantaneous uniform mixing? C_9?
- Conjecture [Mullin 2013]: If a graph admits instantaneous uniform mixing at time τ, then $e^{i\tau}$ must be a root of unity.

  Confirmed for integral Cayley graphs [Cao et al. 2021]
- Conjecture [Mullin 2013]: No connected Cayley graphs on $\mathbb{Z}_n^d$, $n \geq 5$, admits instantaneous uniform mixing.

  Confirmed for integral Cayley graphs [Cao et al. 2021]
- Characterize the connection sets of the Cayley graphs on $\mathbb{Z}_2^d$ that has perfect state transfer or uniform mixing.

  Characterizations of C for perfect state transfer at times $\frac{\pi}{2}$ and $\frac{\pi}{4}$ are known (Cheung and Godsil, 2011)
- Characterize the graphs admitting instantaneous uniform mixing.
- Find complex Hadamard matrices in association schemes.
  Check if instantaneous uniform mixing occurs in any graphs in these schemes.
- Classify the complex Hadamard matrices in the binary Hamming schemes.
- Determine the Nomura algebras of Hadamard matrices of order divisible by 8.
- Find type-II matrices with non-trivial Nomura algebras.
- Find new spin models. (Unitary spin models give mutually unbiased bases.)
- Characterize 3-class association schemes that contain a spin model.

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