Quantum Isomorphisms: Results and Open Problems

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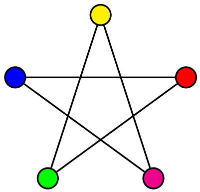
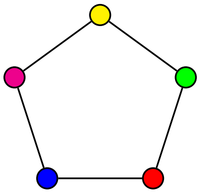
Technical University of Denmark

Open Problems in Algebraic Combinatorics
May 5, 2021

Graph isomorphism



$\cong$



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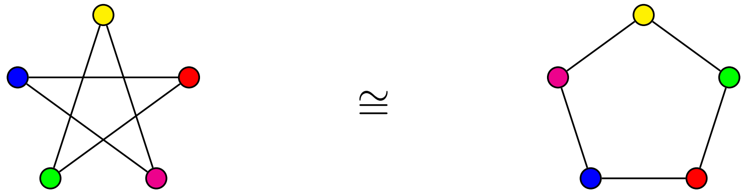

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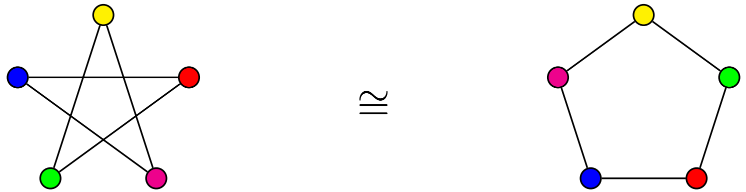

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Matrix formulation: $P^T A_G P = A_H$ for permutation matrix P,
or $A_G P = P A_H$.

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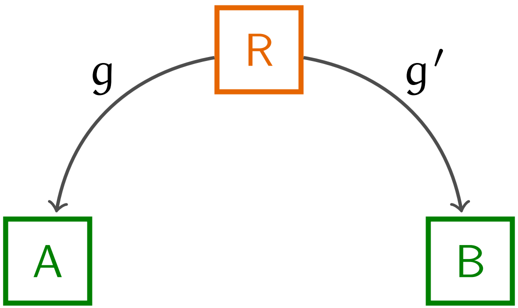

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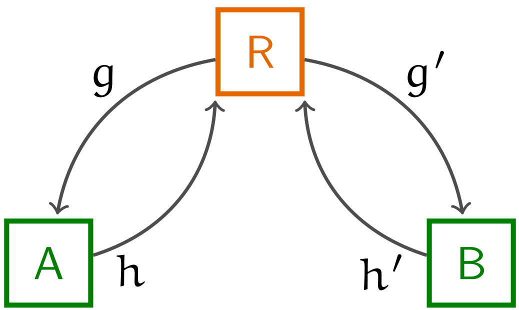

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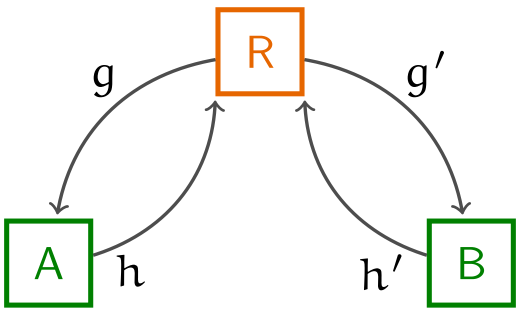

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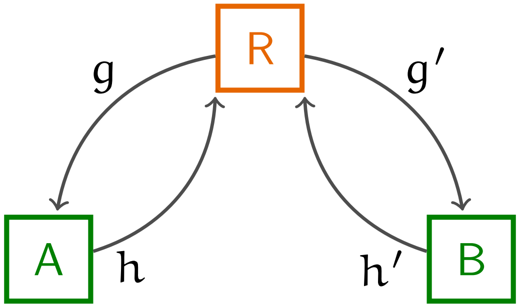

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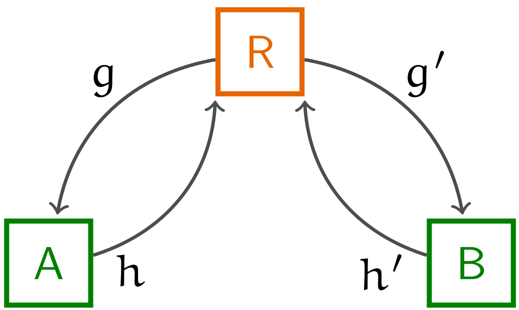

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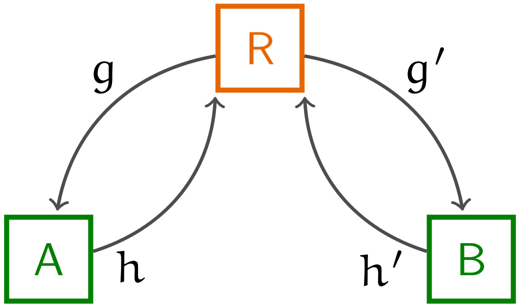

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Proposition. $G \cong H \Leftrightarrow$ Classical players can **win** the game.

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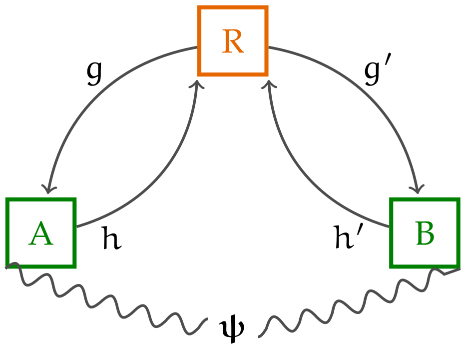

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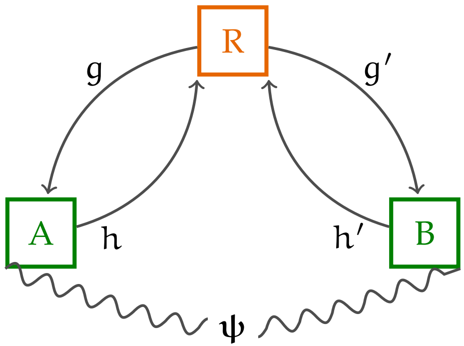

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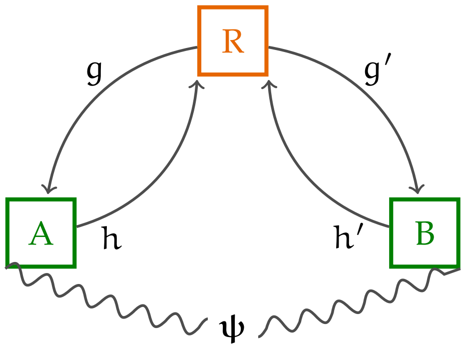

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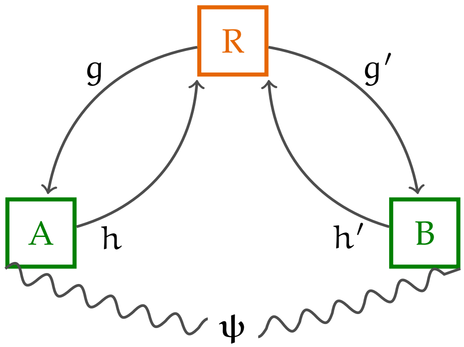

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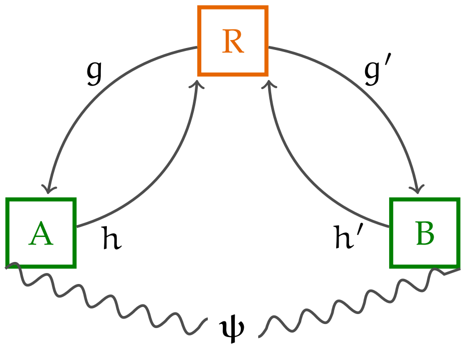

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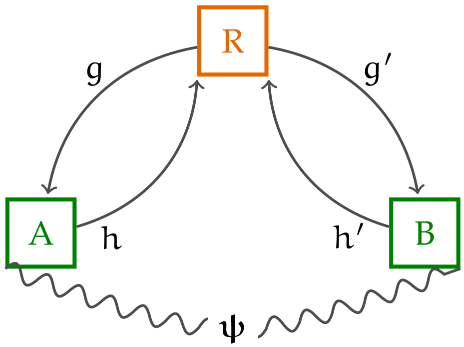

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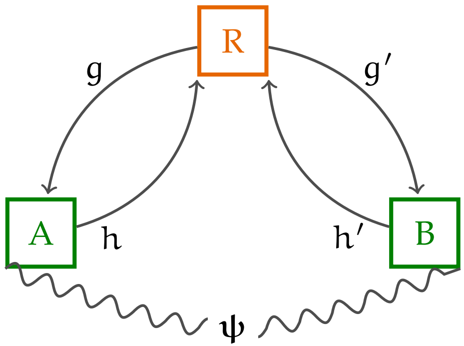

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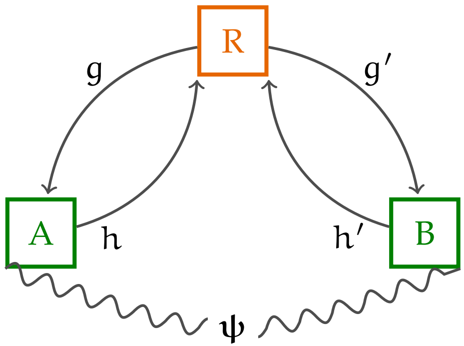

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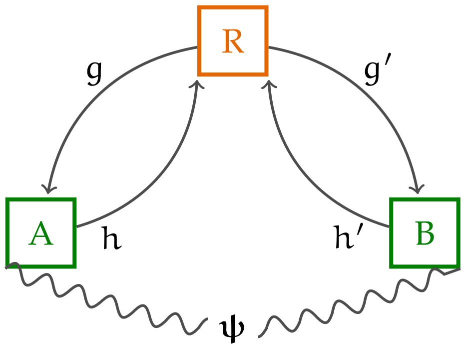

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Definition. G and H are **quantum isomorphic**, denoted $G \cong_{qc} H$ if there is a perfect quantum strategy for the game.

Matrix formulation

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Theorem. (Lupini, Mančinska, R.)
$G \cong_{qc} H$ if and only if there is a QPM $\mathcal{P}$ such that

$$A_G \mathcal{P} = \mathcal{P} A_H$$

$$\text{i.e.} \quad \sum_{g' : g' \sim g} p_{g'h} = \sum_{h' : h' \sim h} p_{gh'}$$

Constructing quantum isomorphic graphs that are not isomorphic

Binary linear systems

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Definition. (Cleve & Mittal, and Cleve, Liu, & Slofstra)
Given $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$, we say that the system $Mx = b$ is **quantum satisfiable** if there are bounded self-adjoint operators $X_1, \ldots, X_m$ on a Hilbert space $\mathcal{H}$ satisfying

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$$x_1 + x_2 + x_3 = 0 \quad \rightarrow \quad X_1 X_2 X_3 = \mathbf{1}$$
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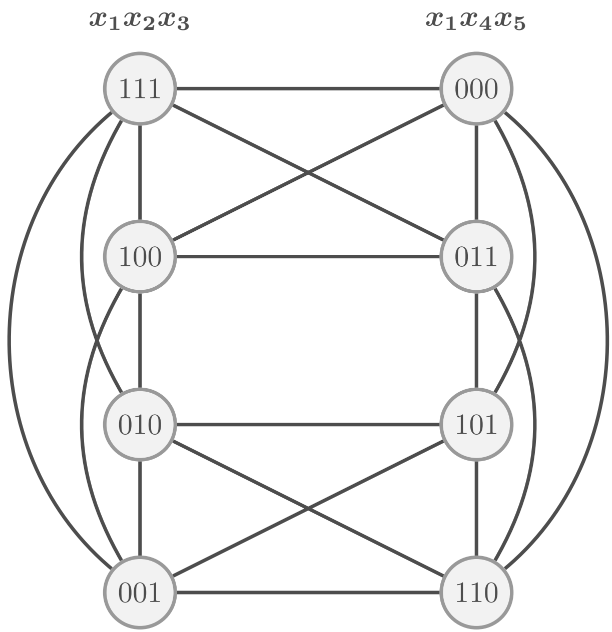
Graph $G(M, b)$:



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Question. How can we produce more examples?

Linear systems from graphs

Definition. The **incidence matrix** M of a graph G is the $V(G) \times E(G)$ matrix such that

$$M_{v,e} = \begin{cases} 1 & \text{if } v \text{ is an endpoint of } e \\ 0 & \text{otherwise} \end{cases}$$

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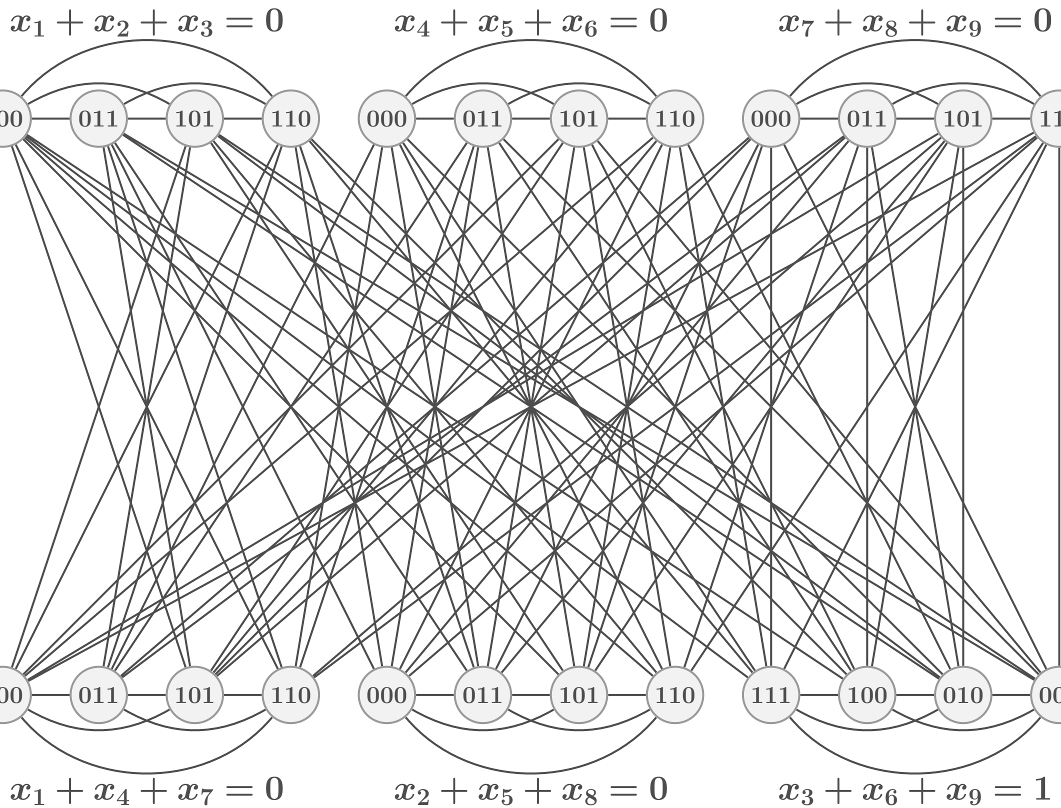
Theorem. (Arkhipov) Let G be a connected graph with incidence matrix M. If $b \in \mathbb{F}_2^{V(G)}$ has **odd weight**, then

1. $Mx = b$ is not satisfiable;
2. $Mx = b$ is quantum satisfiable if and only if G is *not* planar.

Non-isomorphic graphs which are quantum isomorphic



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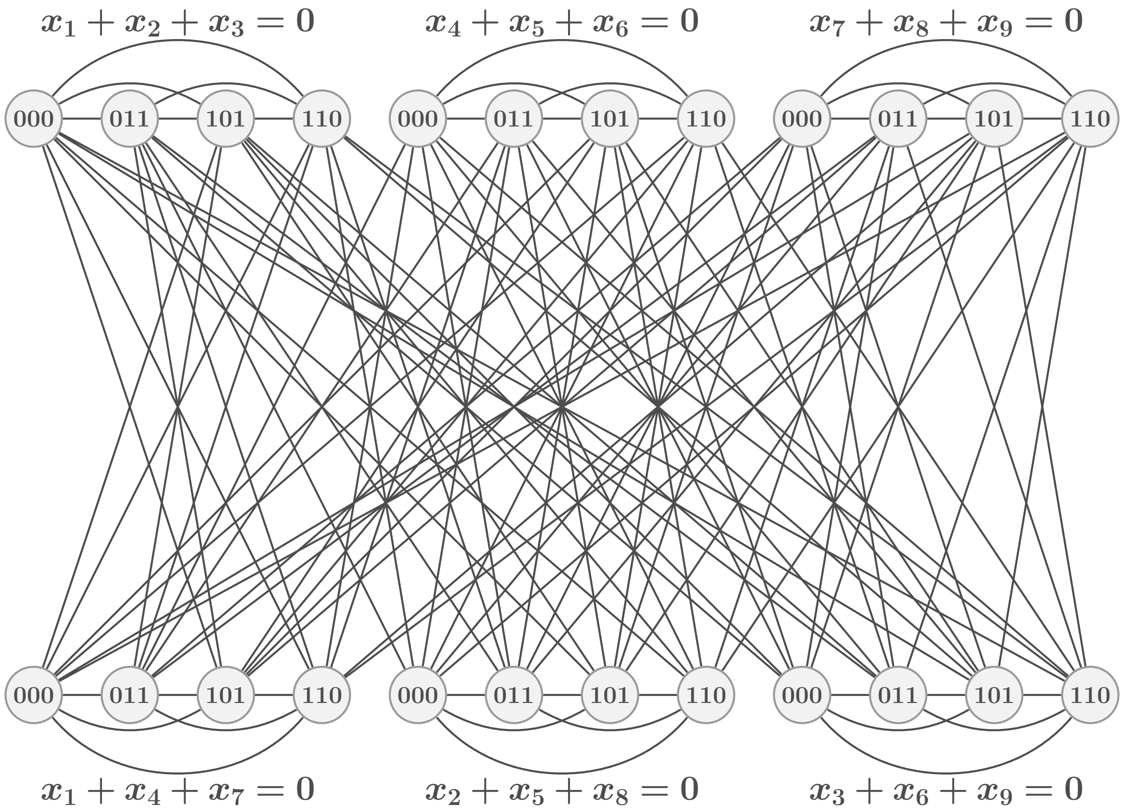

Homomorphism Counting

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Corollary. Given G and H, determining if there is a planar graph F with $\hom(F, G) \neq \hom(F, H)$ is an **undecidable** problem.

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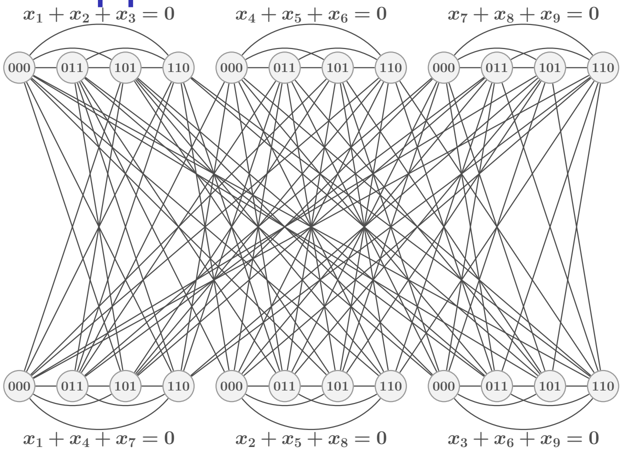
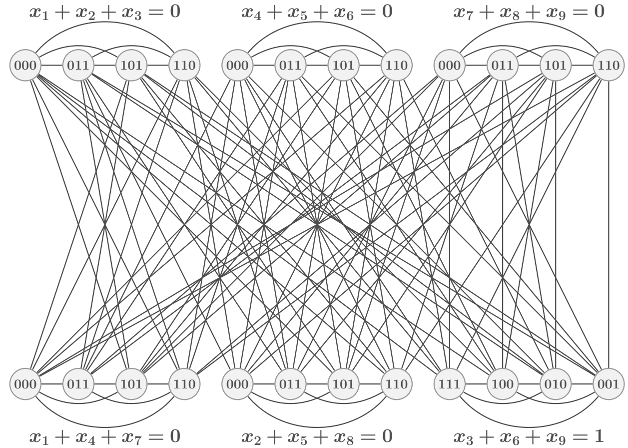
Question. Is there a class $\mathcal{F}$ strictly between planar and all graphs such that $\cong_{\mathcal{F}}$ is tractable?

Cute Application

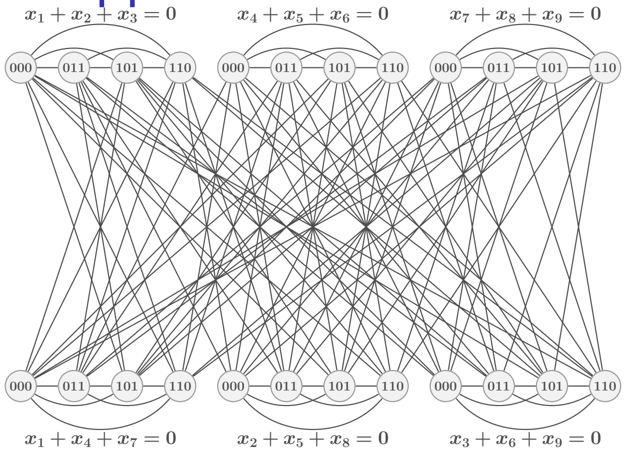


G

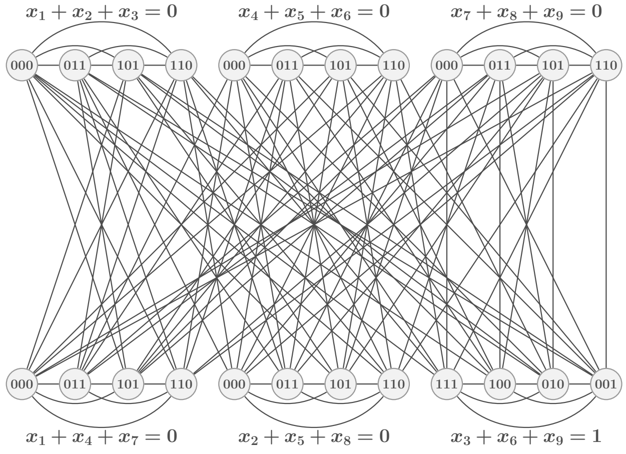


H

Cute Application



G



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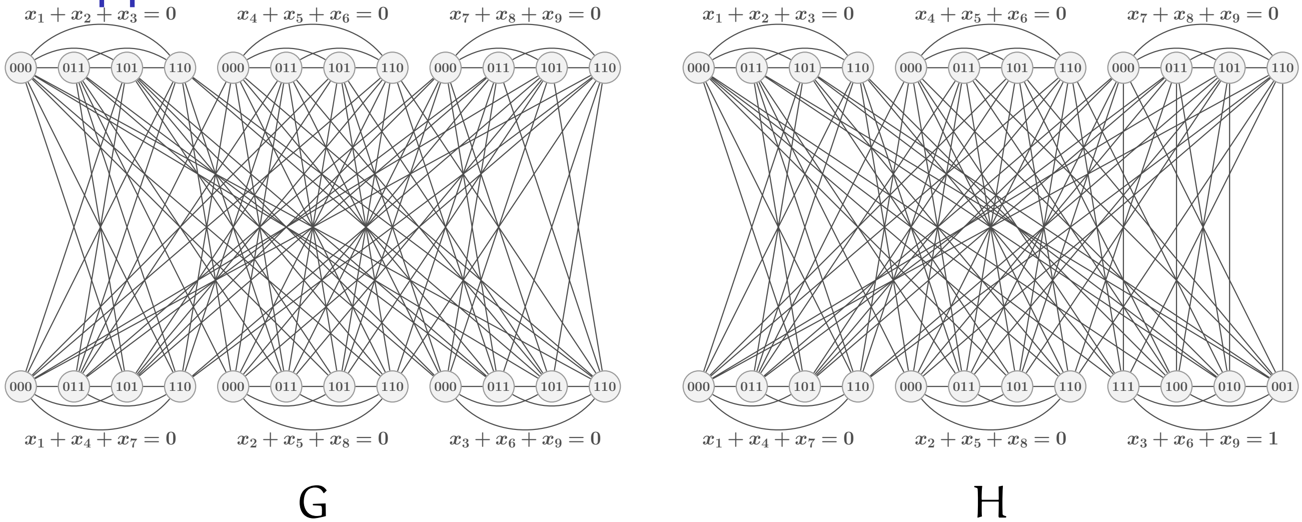

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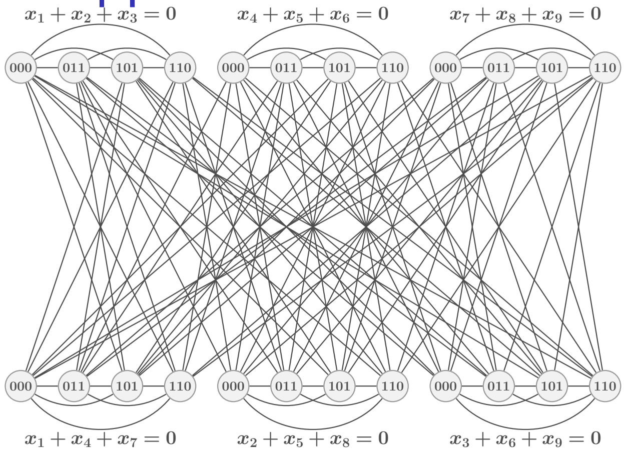

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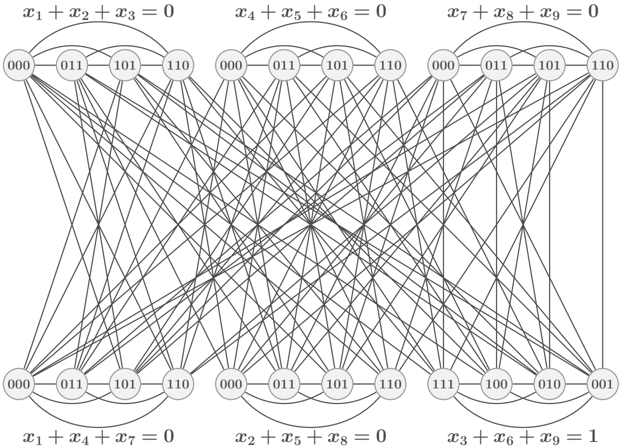

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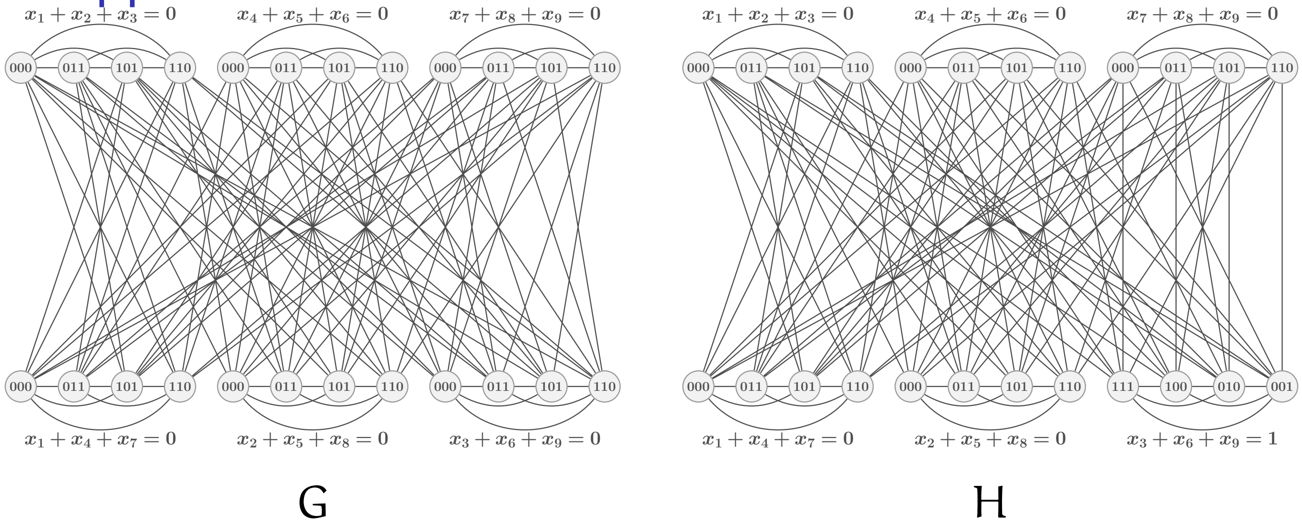

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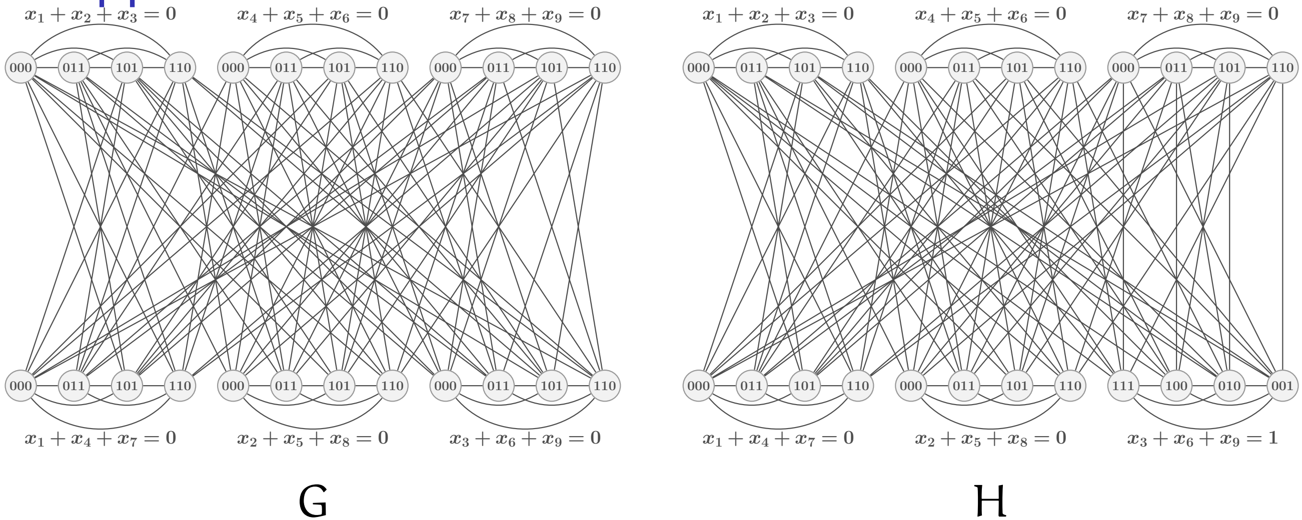

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Question. Can we obtain a useful/novel reformulation of the 4CT this way?

Counting Homomorphisms Part II

A Construction

$E(v) :=$ edges incident to the vertex v.

Definition. Let G be a graph and $U \subseteq V(G)$. Define G_U to be the graph with

$$V(G_U) = \{(v, S) : v \in V(G),\ S \subseteq E(v) \text{ with parity } |\{v\} \cap U|\}$$

such that $(v, S) \sim (u, T)$ if $uv \in E(G)$ and $uv \notin S \triangle T$.

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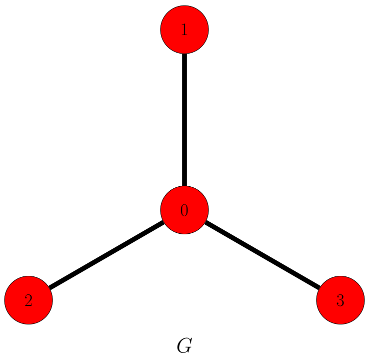
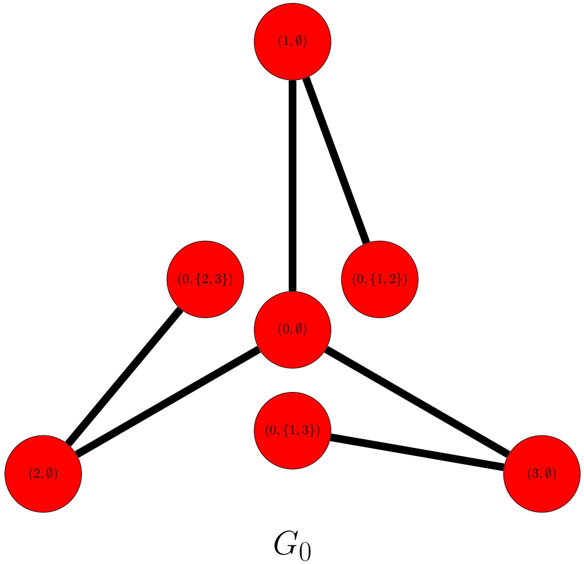
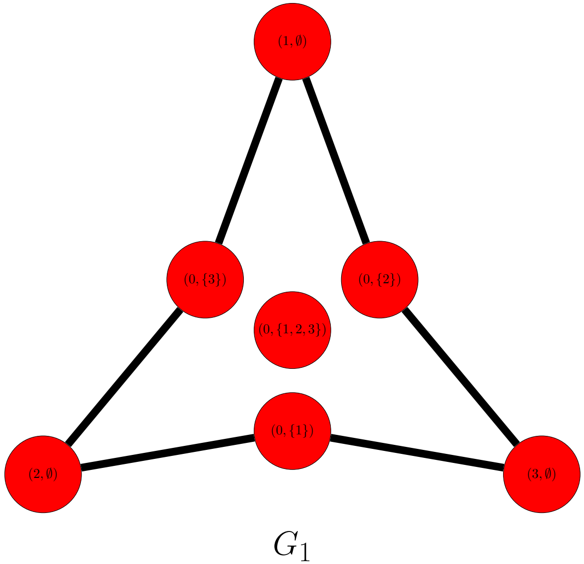
Lemma. If G is a connected graph, then $G_U \cong G_{U'}$ if and only if $|U| \equiv |U'| \mod 2$.

Thus we let $G_0 := G_\varnothing$ and use G_1 to denote the graph isomorphic to $G_{\{v\}}$ for $v \in V(G)$.

Example







Example 2

$G = K_4$

$G_0 =$ the 4×4 Rook graph

$G_1 =$ the Shrikhande graph

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Definition. We denote the **set** of homomorphisms from F to G by $\mathrm{Hom}(F, G)$. Then for any $\psi \in \mathrm{Hom}(F, G)$, we define

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3. $\hom(F, G_0) \geqslant \hom(F, G_1)$ with **strict inequality** if and only if $|\mathrm{Hom}_\psi(F, G_1)| = 0$ for some $\psi \in \mathrm{Hom}(F, G)$.

The Fredholm Alternative

Lemma. $Mx = b$ does not have a solution if and only if

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Corollary. $\hom(F, G_0) \neq \hom(F, G_1)$ if and only if there is $\psi \in \mathrm{Hom}(F, G)$ such that

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Odd Couples

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Definition. Let G be connected, H a graph, and $\psi \in \mathrm{Hom}(H, G)$.

1. $a \in V(H)$ is **odd/even** if $|N_H(a) \cap \psi^{-1}(u)|$ is odd/even $\forall\, u \sim_G \psi(a)$.
2. Denote by V_1 (resp. V_0) the set of odd (resp. even) vertices.
3. (H, ψ) is an **odd couple for G** if $V(H) = V_0 \cup V_1$ and $|V_1 \cap \psi^{-1}(u)|$ is odd for all $u \in V(G)$.

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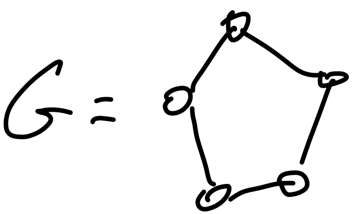

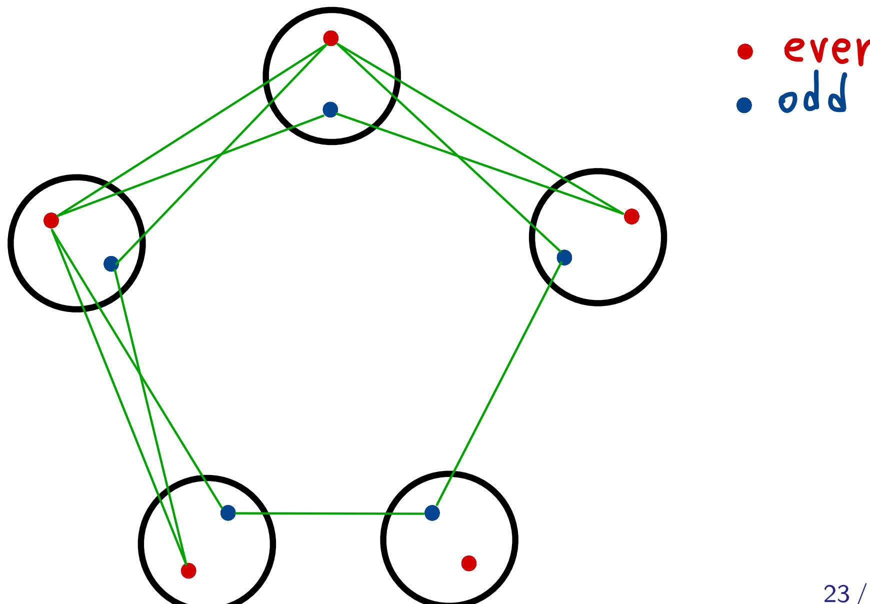

Main Result

Theorem. Let G be a connected graph. Then $\hom(F, G_0) \neq \hom(F, G_1)$ if and only if there exists $\psi \in \mathrm{Hom}(F, G)$ and a subgraph H of F such that $(H, \psi|_H)$ is an odd couple for G.

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Remark. If (H, ψ) is an odd couple for G, then $\Delta(H) \geqslant \Delta(G)$. Therefore, if $\hom(F, G_0) \neq \hom(F, G_1)$, then $\Delta(F) \geqslant \Delta(G)$.

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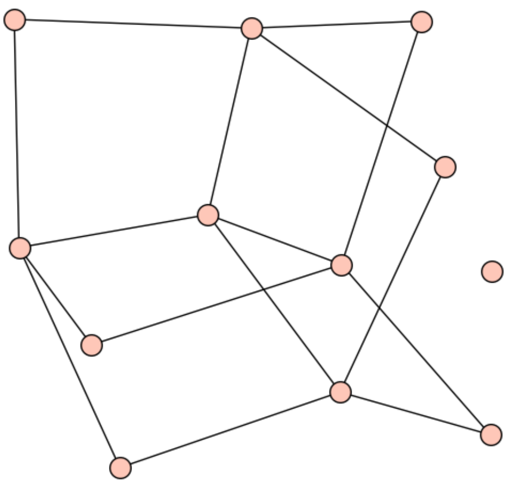
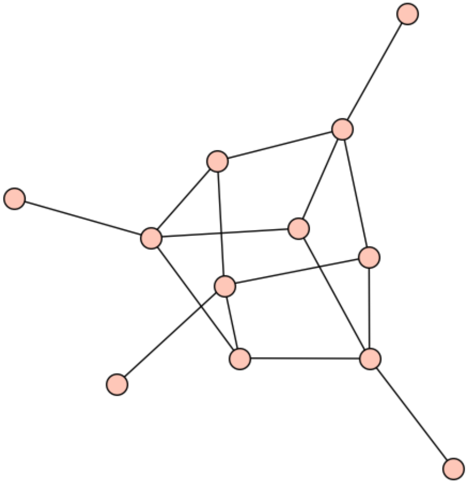
Corollary. If $\mathcal{F}$ is the family of graphs F with $\Delta(F) \leqslant d$, then $G_0 \cong_{\mathcal{F}} G_1$ for $G = K_{1,d+1}$.

$G = K_{1,4}$



G_0



G_1

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- **Bonus:** What is the convex hull of $\{P \otimes P : P \text{ a permutation}\}$?

Thank you!!

Quantum Groups

The quantum automorphism group of a graph G

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Definition. (Banica)
$C(\mathrm{Qut}(G))$ is the universal C^*-algebra generated by elements p_{ij} satisfying the following:

1. $p_{ij} = p_{ij}^2 = p_{ij}^*$ for all i, j;
2. $\sum_k p_{ik} = \mathbf{1} = \sum_\ell p_{\ell j}$ for all i, j;

} $\mathcal{P} = (p_{ij})$ is a QPM.

3. $\sum_{k \sim i} p_{kj} = \sum_{\ell \sim j} p_{i\ell}$ for all i, j. — $A_G \mathcal{P} = \mathcal{P} A_G$.

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Remark. The matrix $\mathcal{P}$ is called the **fundamental representation** of $\mathrm{Qut}(G)$.

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Theorem.
If G and H are connected, then $G \cong_{qc} H$ if and only if there exist $g \in V(G)$ and $h \in V(H)$ in the same orbit of $\mathrm{Qut}(G \cup H)$.

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Lemma. (Lupini, Mančinska, R)

1. $T \in \mathcal{T}(1, 0) \Leftrightarrow T_i = T_j$ whenever $i \sim_1 j$;
2. $T \in \mathcal{T}(1, 1) \Leftrightarrow T_{i,\ell} = T_{j,k}$ whenever $(i, \ell) \sim_2 (j, k)$.

Intertwiners

$$\left(\mathcal{P}^{\otimes k}\right)_{i_1 i_2 \dots i_k, j_1 j_2 \dots j_k} = p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_k j_k}$$

Definition. (ℓ, k)-intertwiner of $\mathrm{Qut}(G)$

$$\mathcal{T}(\ell, k) := \left\{ T \in \mathbb{C}^{V(G)^\ell \times V(G)^k} : \mathcal{P}^{\otimes \ell} T = T \mathcal{P}^{\otimes k} \right\}.$$

Lemma. (Lupini, Mančinska, R)

1. $T \in \mathcal{T}(1,0) \Leftrightarrow T_i = T_j$ whenever $i \sim_1 j$;
2. $T \in \mathcal{T}(1,1) \Leftrightarrow T_{i,\ell} = T_{j,k}$ whenever $(i,\ell) \sim_2 (j,k)$.

Intertwiners of $\mathrm{Qut}(G) = \langle U, M, A_G \rangle_{\circ, \otimes, *, \mathrm{lin}}$, where

$$U = \sum_{i \in V(G)} e_i, \quad M(e_i \otimes e_j) = \delta_{ij} e_i \;\; \forall i, j \in V(G).$$

Bi-labeled graphs

Definition. (Lovász, Large Networks and Graph Limits)
An (ℓ, k)-**bi-labeled graph** is a triple $\vec{F} = (F, \vec{a}, \vec{b})$ where

- F is a graph;
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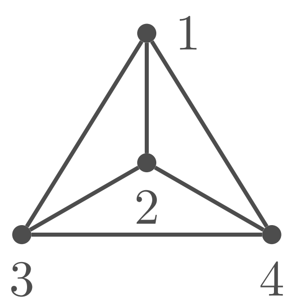
Example. $\vec{F} = \big(K_4, (2,1), (2,2)\big)$.



K_4

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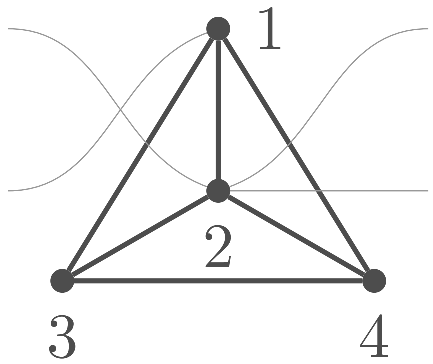

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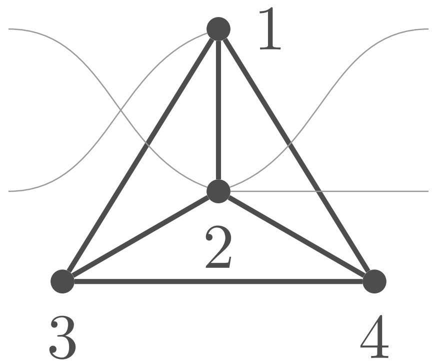
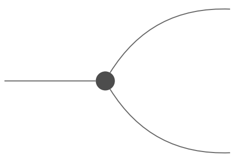


$\vec{U}$



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Let G be a graph and $\vec{F} = (F, (a), (b))$ an $(1, 1)$-bi-labeled graph.

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Theorem. For a graph G and bi-labeled graphs $\vec{F}_1, \vec{F}_2$,

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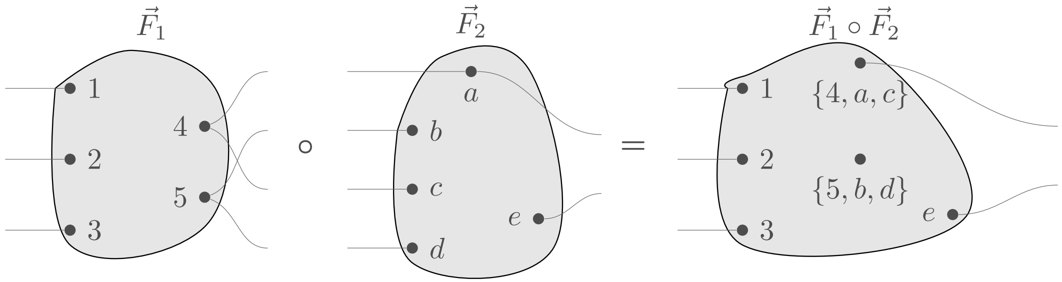

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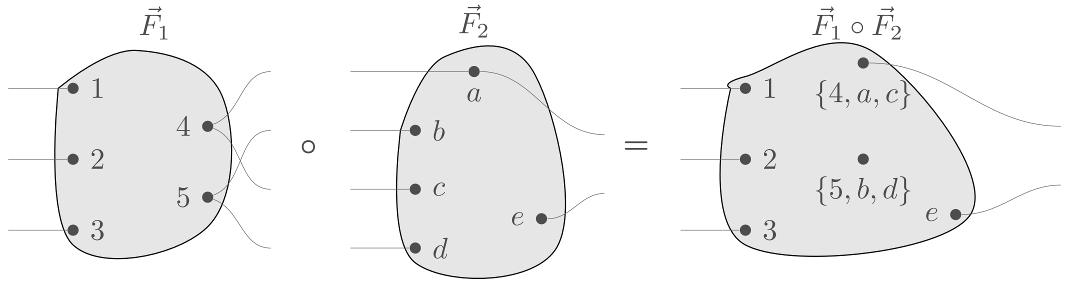

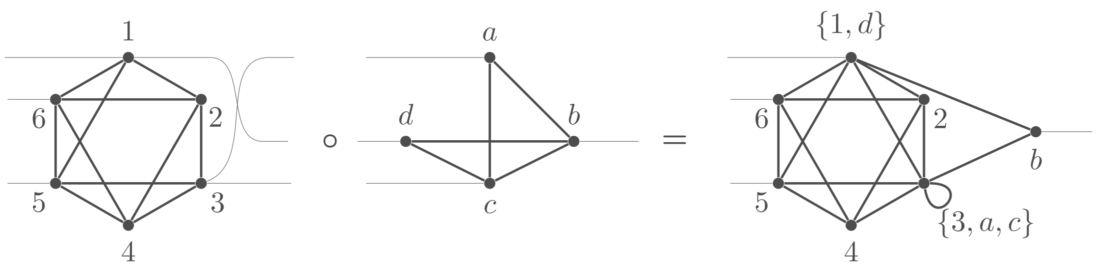

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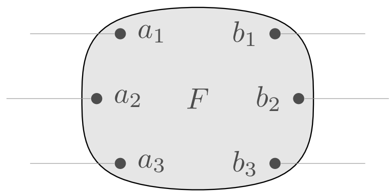

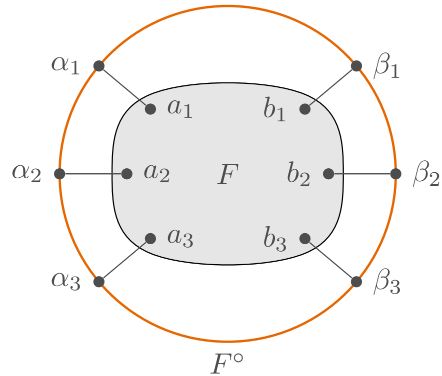

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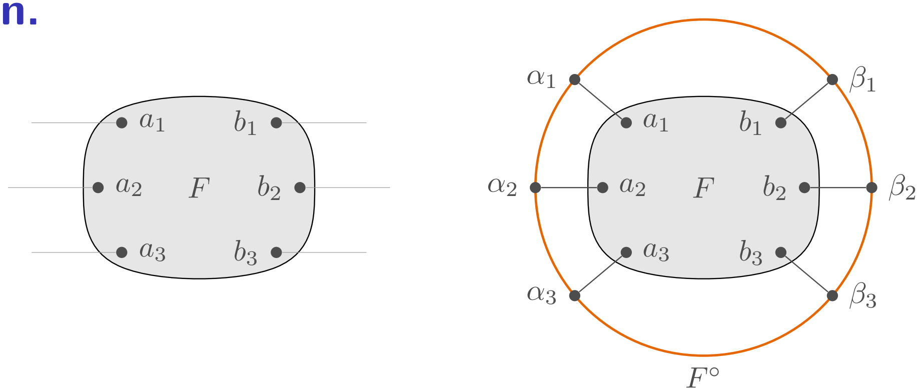

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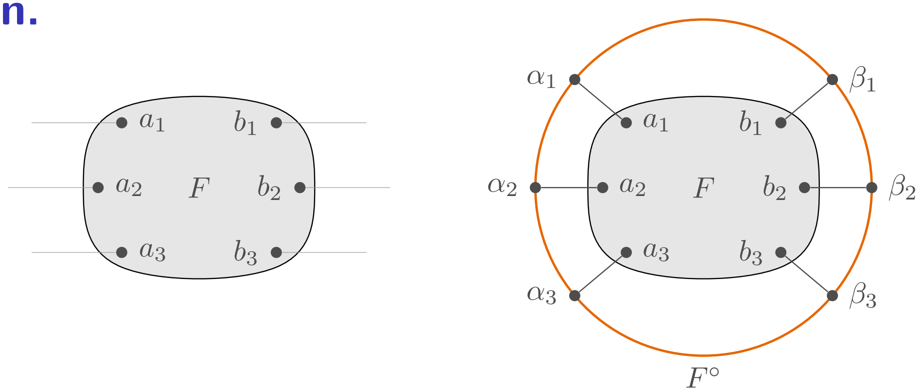

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Theorem. (Mančinska & R)

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