Equiangular lines over finite fields – Problem list

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Open Problems in Algebraic Combinatorics
Online Workshop
May 4, 2021

Major open problems

Problem (Relative bound)

For which (d, n) is there a real $d \times n$ ETF? Complex?

- ▶ Real case: ETFs are equivalent to SRGs with $k = 2\mu$, many necessary conditions, smallest open case is ~~33×66~~ 43×86
- ▶ Complex case: Many examples, but nonexistence is hard

Problem (Real absolute bound)

For which d is there a real $d \times \binom{d+1}{2}$ ETF?

- ▶ Existence known only for $d = 2, 3, 7, 23$
- ▶ Necessary conditions: $d \leq 3$ or $d + 2$ is an odd square
- ▶ Next open case is $d = 79$

Major open problems

> ## Problem (Complex absolute bound / Zauner's conjecture)
> Prove that for every $d \geq 1$, there is a complex $d \times d^2$ ETF.

- ▶ Numerical evidence for $d \leq 151$ (then computers are slow)
- ▶ Known for only finitely many dimensions d (e.g. $d \leq 24$)
- ▶ 2021 EUR prize for proof in infinitely many dimensions
- ▶ Seems related to Stark conjectures and Hilbert's 12th problem

Zauner, PhD Thesis, U Vienna, 1999

Golden KCIK Award, arXiv:2002.03233

Appleby, Flammia, McConnell, Yard, Found. Phys., 2017

Kopp, Int. Math. Res. Not., 2019

Image from gerhardzauner.at

More open problems

Problem

Does every SRG arise from some kind of "ETF"?

- ▶ Brouwer's table of feasible SRG parameters: 211 known complementary pairs of parameters with $v \leq 1300$
- ▶ All but 9 pairs (95%) come from finite field ETFs:

v	k	λ	μ	v	k	λ	μ	v	k	λ	μ
21	10	3	6	70	27	12	9	220	84	38	28
40	12	2	4	112	30	2	10	280	117	44	52
57	24	11	9	120	42	8	18	512	196	60	84

More open problems

Problem (Relative bound)

Given an orthogonal geometry on $\mathbb{F}_q^d$ and $a, b \in \mathbb{F}_q$, find an upper bound for the size of an (a, b)-equiangular system.

- ▶ Efficient bound may disprove SRGs
- ▶ Is there a $(2, 1)$-equiangular system with 100 vectors in an orthogonal geometry on $\mathbb{F}_5^{45}$?
- ▶ If not, then Conway's 99-graph DNE

Problem

Suppose that, for infinitely many pairwise coprime q, there is a $d \times n$ ETF in a unitary geometry on $\mathbb{F}_{q^2}^d$. Is there $d \times n$ complex ETF as well?

More open problems

Problem

For which (q, d) is Gerzon's bound $n \leq d^2$ saturated by an (a, b)-equiangular system with $a^2 \neq b$ in a unitary geometry on $\mathbb{F}_{q^2}^d$?

- ▶ Zauner: For every d, there exists q
- ▶ Harder than Zauner to solve completely
- ▶ Easier to make progress

Problem

Generalize (doubly transitive) complex 3×9 to an infinite family over a finite field.

More problems

Problem

Is there a combinatorial description of ETFs in finite unitary geometry?

- ▶ ETFs in orthogonal geometry are equivalent to "modular SRGs"

Problem

Find necessary conditions (e.g. integrality constraints) for ETF existence in unitary geometry.

- ▶ Efficient solution may give necessary conditions for existence of complex ETFs