Equiangular lines over finite fields

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Outline

Context: Real and complex equiangular lines

Equiangular lines in orthogonal geometry

Equiangular lines in unitary geometry

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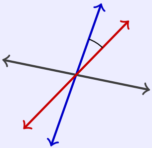
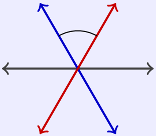
Line packing

Problem

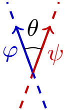
Pack n lines (1-dim subspaces) in $\mathbb{R}^d$ or $\mathbb{C}^d$ without sharp angles



Bad



Good



$\cos\theta = |\langle \varphi, \psi \rangle|$

- Given lines, choose unit norm reps

$$\Phi = \begin{bmatrix} | & & | \\ \varphi_1 & \cdots & \varphi_n \\ | & & | \end{bmatrix} \in \mathbb{C}^{d \times n}$$

- To avoid sharp angles, minimize **coherence**

$$\mu = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|$$

An application

Uniquely solve an underdetermined system

$$\begin{bmatrix} & \Phi & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} y \end{bmatrix}$$

Image from `mylittledrummerboys.blogspot.com`

Donoho & Elad, Proc. Natl. Acad. Sci. USA, 2003

Candès, Romberg & Tao, IEEE Trans. Inform. Theory, 2006

An application

Uniquely solve an underdetermined system

$$\begin{bmatrix} & \Phi & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} y \end{bmatrix}$$

Compressed sensing

$\Phi x = y$ has a unique **sparse** solution if

- x has a lot of zero entries, and
- the columns of Φ are "very different"

In fact, x can be found by a linear program

- Span lines with wide angles $\implies$ "very different"
- Better coherence $\implies$ fewer zeros required

Image from `mylittledrummerboys.blogspot.com`

Donoho & Elad, Proc. Natl. Acad. Sci. USA, 2003

Candès, Romberg & Tao, IEEE Trans. Inform. Theory, 2006

Recognizing optimal packings

Theorem (Welch bound)

Given n unit vectors spanning $\mathbb{R}^d$ or $\mathbb{C}^d$

$$\Phi = \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} \in \mathbb{C}^{d\times n},$$

their coherence $\mu := \max_{i\neq j} |\langle \varphi_i, \varphi_j \rangle|$ satisfies

$$\mu \geq \sqrt{\frac{n-d}{d(n-1)}}.$$

Equality holds iff Φ is an **equiangular tight frame** (ETF):

- Equiangular: $|\langle \varphi_i, \varphi_j \rangle| = \mu$ for all $i \neq j$
- Tight frame: $\Phi\Phi^* = \mathrm{const} \cdot I$

Proof.

$0 \leq \|\Phi\Phi^* - \frac{n}{d} I_d\|_F^2 = \sum_{i,j\in[n]} |\langle \varphi_i, \varphi_j \rangle|^2 - \frac{n^2}{d} \leq (n^2-n)\mu^2 + n - \frac{n^2}{d}$

Welch, IEEE Trans. Inform. Theory, 1974

Image from calit2.net

Real and complex equiangular lines

Corollary (Relative bound)

Suppose $\mu^2 < \frac{1}{d}$ and there are n unit vectors

$$\Phi = \begin{bmatrix} \varphi_1 & \dots & \varphi_n \end{bmatrix}$$

in $\mathbb{R}^d$ or $\mathbb{C}^d$ such that $|\langle \varphi_i, \varphi_j \rangle| = \mu$ for every $i \neq j$. Then

$$n \leq \frac{d(1-\mu^2)}{1-d\mu^2}.$$

Equality holds if and only if Φ is an ETF.

J.J. Seidel

Lemmens and Seidel, J. Algebra, 1973

Image from "Geometry and Combinatorics", Academic Press, 1991

Real and complex equiangular lines

Theorem (Absolute/Gerzon bound)

If there are n unit vectors

$$\Phi = \begin{bmatrix} \varphi_1 & \dots & \varphi_n \end{bmatrix}$$

in $\mathbb{R}^d$ or $\mathbb{C}^d$ such that $|\langle \varphi_i, \varphi_j \rangle| = \mu$ is constant for $i \neq j$, then

$$n \leq \begin{cases} \binom{d+1}{2}, & \text{real case;} \\ d^2, & \text{complex case.} \end{cases}$$

If equality holds, then Φ is an ETF.

Michael Gerzon

Proof of bound.

- $\left[\langle \varphi_i \varphi_i^*, \varphi_j \varphi_j^* \rangle_F \right]_{ij} = \left[|\langle \varphi_i, \varphi_j \rangle|^2 \right]_{ij} = \mu J + (1-\mu) I$ is invertible
- So $\{\varphi_j \varphi_j^*\}_{j \in [n]}$ is linearly independent in the real space of self-adjoint matrices

Lemmens and Seidel, J. Algebra, 1973

Image from `michaelgerzonphotos.org.uk/`

Major open problems

Problem (Relative bound)

For which (d, n) is there a real $d \times n$ ETF? Complex?

- Real case: ETFs are equivalent to SRGs with $k = 2\mu$, many necessary conditions, smallest open case is $\cancel{33 \times 66}$ 43×86
- Complex case: Many examples, but nonexistence is hard

Major open problems

Problem (Relative bound)

For which (d, n) is there a real $d \times n$ ETF? Complex?

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- Complex case: Many examples, but nonexistence is hard

Problem (Real absolute bound)

For which d is there a real $d \times \binom{d+1}{2}$ ETF?

- Existence known only for $d = 2, 3, 7, 23$
- Necessary conditions: $d \leq 3$ or $d + 2$ is an odd square
- Next open case is $d = 79$

Major open problems

Problem (Complex absolute bound / Zauner's conjecture)

Prove that for every $d \geq 1$, there is a complex $d \times d^2$ ETF.

Gerhard Zauner

- Numerical evidence for $d \leq 151$ (then computers are slow)
- Known for only finitely many dimensions d (e.g. $d \leq 24$)
- 2021 EUR prize for proof in infinitely many dimensions
- Seems related to Stark conjectures and Hilbert's 12th problem

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Zauner, PhD Thesis, U Vienna, 1999

Golden KCIK Award, `arXiv:2002.03233`

Appleby, Flammia, McConnell, Yard, Found. Phys., 2017

Kopp, Int. Math. Res. Not., 2019

Image from `gerhardzauner.at`

This talk

Our approach

Consider finite field versions of the above problems

- Orthogonal geometry $\approx$ finite $\mathbb{R}^d$
- Unitary geometry $\approx$ finite $\mathbb{C}^d$

What's coming:

- Basic definitions and results
- Interactions with real/complex cases
- Connections with algebraic combinatorics
- Gerzon equality in infinitely many dimensions for both orthogonal and unitary geometry

Outline

Context: Real and complex equiangular lines

Equiangular lines in orthogonal geometry

Equiangular lines in unitary geometry

Finite orthogonal geometry

Throughout this section, $q = p^k$ is an odd prime power.

Definition

An **orthogonal geometry** on $\mathbb{F}_q^d$ is given by a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle \colon \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$. That means for every x and y:

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, \cdot \rangle \colon \mathbb{F}_q^d \to \mathbb{F}_q$ is linear
- if $x \neq 0$, then $\langle x, \cdot \rangle \colon \mathbb{F}_q^d \to \mathbb{F}_q$ is not the zero mapping

Examples

- $\langle x, y \rangle = x^\top y$
- $M = M^\top \in \mathbb{F}_q^{d \times d}$ is invertible, and $\langle x, y \rangle = x^\top M y$
- Every example takes this form with $M = \left[\langle e_i, e_j \rangle\right]_{ij}$

Two kinds of orthogonal geometry

Notation:

- $\mathbb{F}_q^\times$ is the multiplicative group of units
- $\mathbb{F}_q^{\times 2} \leq \mathbb{F}_q^\times$ is the subgroup of quadratic residues (squares)

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Change of basis:

- If $\langle x, y \rangle = x^\top My$ and if $B \in \mathbb{F}_q^{d \times d}$ is invertible, then $\langle Bx, By \rangle = x^\top B^\top MBy$

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- $\det(B^\top M B) = \det(M) \det(B)^2 \in \det(M) \, \mathbb{F}_q^{\times 2}$
- $\det(M) \, \mathbb{F}_q^{\times 2} \in \mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}$ is an invariant of $\langle \cdot, \cdot \rangle$

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Change of basis:

- If $\langle x, y\rangle = x^\top My$ and if $B \in \mathbb{F}_q^{d\times d}$ is invertible, then $\langle Bx, By\rangle = x^\top B^\top MBy$
- $\det(B^\top MB) = \det(M)\det(B)^2 \in \det(M)\,\mathbb{F}_q^{\times 2}$
- $\det(M)\,\mathbb{F}_q^{\times 2} \in \mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ is an invariant of $\langle\cdot,\cdot\rangle$

Proposition

Up to linear isometry, there exactly two kinds of orthogonal geometry on $\mathbb{F}_q^d$:

- *Square type:* $\det M \in \mathbb{F}_q^{\times 2}$ *(e.g. $\langle x, y\rangle = x^\top y$)*
- *Nonsquare type:* $\det M \notin \mathbb{F}_q^{\times 2}$

Equiangular systems and tight frames

Fix an orthogonal geometry on $\mathbb{F}_q^d$

Definition

For $a, b, c \in \mathbb{F}_q$, a sequence $\varphi_1, \ldots, \varphi_n \in \mathbb{F}_q^d$ is:

- a **frame** if it spans $\mathbb{F}_q^d$;

Equiangular systems and tight frames

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- a **frame** if it spans $\mathbb{F}_q^d$;
- a c-**tight frame** if it spans $\mathbb{F}_q^d$ and moreover

$$\sum_{j \in [n]} \langle \varphi_j, x \rangle \varphi_j = cx$$

  for every $x \in \mathbb{F}_q^d$;

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$$\textstyle\sum_{j \in [n]} \langle \varphi_j, x \rangle \varphi_j = cx$$
for every $x \in \mathbb{F}_q^d$;
- an (a, b)-**equiangular system** if:
  (i) $\langle \varphi_i, \varphi_i \rangle = a$ for every $i \in [n]$, and
  (ii) $\langle \varphi_i, \varphi_j \rangle^2 = b$ for every $i \neq j$ in $[n]$;

Equiangular systems and tight frames

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for every $x \in \mathbb{F}_q^d$;
- an (a, b)-**equiangular system** if:
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  (ii) $\langle \varphi_i, \varphi_j\rangle^2 = b$ for every $i \neq j$ in $[n]$;
- an (a, b, c)-**equiangular tight frame** (ETF) if it is an (a, b)-equiangular system and a c-tight frame.

Differences from real case

Examples of differences with $p = 3$ and $\langle x, y \rangle = x^\top y$

Nonzero vectors can have "norm" 0:

- $x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ has $\langle x, x \rangle = 0$

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- $\Phi = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ has $\Phi\Phi^\top = 0I$

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Spectral theorem fails:

- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^2 = 0$, so $\sigma(J_3) = \{0^2\}$ and J_3 is not diagonalizable

Example: $4 \times 10 = \binom{4+1}{2}$ ETF

$p = 3,\ M = \operatorname{diag}(1,1,1,2)$

$$\Phi = \begin{bmatrix} 1 & 2 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{F}_3^{4\times 10}$$

is an ETF with $(a,b,c) = (0,1,0)$ since $\operatorname{rank}\Phi = 4$, $\Phi\Phi^\top M = 0$,

$$\big[\langle \varphi_i, \varphi_j \rangle\big]_{ij} = \begin{bmatrix} 0 & - & + & + & - & - & + & + & + & + \\ - & 0 & - & + & + & + & - & + & + & + \\ + & - & 0 & - & + & + & + & - & + & + \\ + & + & - & 0 & - & + & + & + & - & + \\ - & + & + & - & 0 & + & + & + & + & - \\ - & + & + & + & + & 0 & + & - & - & + \\ + & - & + & + & + & + & 0 & + & - & - \\ + & + & - & + & + & - & + & 0 & + & - \\ + & + & + & - & + & - & - & + & 0 & + \\ + & + & + & + & - & + & - & - & + & 0 \end{bmatrix}$$

Image from `wikimedia.org`

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$p = 3$, $M = \mathrm{diag}(1,1,1,2)$

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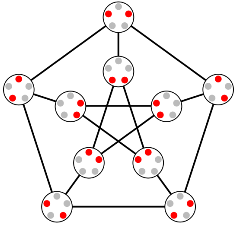

Image from `wikimedia.org`

Frames from Gram matrices

Suppose $\langle x, y \rangle = x^\top M y$.

If $\varphi_1, \ldots, \varphi_n \in \mathbb{F}_q^d$ spans, then it contains a basis, and a principal submatrix of $\left[\langle \varphi_i, \varphi_j \rangle\right]_{ij}$ is a Gram matrix for $\langle \cdot, \cdot \rangle$.

Frames from Gram matrices

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If $\varphi_1, \ldots, \varphi_n \in \mathbb{F}_q^d$ spans, then it contains a basis, and a principal submatrix of $\left[\langle\varphi_i, \varphi_j\rangle\right]_{ij}$ is a Gram matrix for $\langle\cdot,\cdot\rangle$.

Theorem

Given $G \in \mathbb{F}_q^{n\times n}$, choose columns that form a basis for Col G*, and let G_b be the corresponding principal submatrix. There exists a spanning set $\varphi_1, \ldots, \varphi_n \in \mathbb{F}_q^d$ such that $G = \left[\langle\varphi_i, \varphi_j\rangle\right]_{ij}$ iff:*

(i) $G^\top = G$,

(ii) $\operatorname{rank} G = d$,

(iii) $\det(G_b) \in \det(M)\,\mathbb{F}_q^{\times 2}$.

- Every symmetric matrix is a Gram matrix in **some** geometry

ETFs from Gram matrices

Corollary

Suppose $G \in \mathbb{F}_q^{n\times n}$ has $\operatorname{rank} G = d$. Then G is the Gram matrix of an (a,b,c)-ETF in an orthogonal geometry on $\mathbb{F}_q^d$ iff:

(i) $G = G^\top$,

(ii) $G_{ii} = a$ for every $i \in [n]$,

(iii) $(G_{ij})^2 = b$ for every $i \neq j$ in $[n]$,

(iv) $G^2 = cG$.

$$G = \begin{bmatrix} 0 & - & + & + & - & - & + & + & + & + \\ - & 0 & - & + & + & + & - & + & + & + \\ + & - & 0 & - & + & + & + & - & + & + \\ + & + & - & 0 & - & + & + & + & - & + \\ - & + & + & - & 0 & + & + & + & + & - \\ - & + & + & + & + & 0 & + & - & - & + \\ + & - & + & + & + & + & 0 & + & - & - \\ + & + & - & + & + & - & + & 0 & + & - \\ + & + & + & - & + & - & - & + & 0 & + \\ + & + & + & + & - & + & - & - & + & 0 \end{bmatrix}, \; G^2 = 0 \qquad \rightsquigarrow \qquad \Phi = \begin{bmatrix} 1 & 2 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$(a,b,c) = (0,1,0)$$

Overlap with the real case

Recall $q = p^k$ is an odd prime power.

Proposition

Suppose there is a real $d \times n$ ETF with $n > d + 1$, and either

(i) $n \neq 2d$ *and* $p \nmid \sqrt{\frac{d(n-1)}{n-d}}$*; or*

(ii) $n = 2d$ *and* $n - 1 \in \mathbb{F}_q^{\times 2}$*.*

Then there is a $d \times n$ ETF in an orthogonal geometry on $\mathbb{F}_q^d$.

Proposition

Suppose there is a $d \times n$ ETF in an orthogonal geometry on $\mathbb{F}_q^d$. If $p > 2n - 5$, then there is a real $d \times n$ ETF.

- Moral: Lots of overlap with the real case
- Caveat: Weird stuff in the cracks!

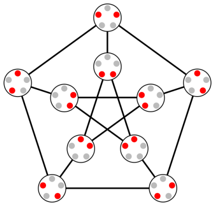
Strongly regular graphs

Definition

A (simple, undirected) graph is **strongly regular** if it is neither edgeless nor complete, and there are parameters k, λ, μ such that:

- every vertex has exactly k neighbors,
- every pair of adjacent vertices have exactly λ neighbors in common, and
- every pair of distinct nonadjacent vertices have exactly μ neighbors in common.

If there are v vertices, then we call it a (v, k, λ, μ)-SRG.



Petersen graph:
$v = 10,\ k = 3,$
$\lambda = 0,\ \mu = 1$

SRG notation

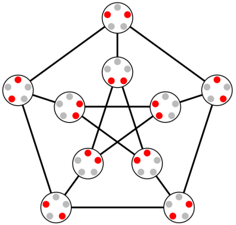
Given a (v, k, λ, μ)-SRG:

- The $\{0, 1\}$-adjacency matrix A has $\sigma(A) =: \{k^1, r^f, s^g\}$, where $r > 0 > s$
- Each of r, s, f, g is a function of (v, k, λ, μ)

SRG notation

Given a (v, k, λ, μ)-SRG:

- The $\{0,1\}$-adjacency matrix A has $\sigma(A) =: \{k^1, r^f, s^g\}$, where $r > 0 > s$
- Each of r, s, f, g is a function of (v, k, λ, μ)
- The **Seidel adjacency matrix** $\Sigma := J - 2A - I$ has $\sigma(\Sigma) = \{(v-2k-1)^1, (-2r-1)^f, (-2s-1)^g\}$
- $\Sigma_{ij} = \begin{cases} -1, & \text{if } i \sim j; \\ 1, & \text{if } i \nsim j \text{ and } i \neq j; \\ 0, & \text{if } i = j \end{cases}$



$$\begin{bmatrix} 0&1&0&0&1&1&0&0&0&0 \\ 1&0&1&0&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0&1&0&0 \\ 0&0&1&0&1&0&0&0&1&0 \\ 1&0&0&1&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&1&1&0 \\ 0&1&0&0&0&0&0&0&1&1 \\ 0&0&1&0&0&1&0&0&0&1 \\ 0&0&0&1&0&1&1&0&0&0 \\ 0&0&0&0&1&0&1&1&0&0 \end{bmatrix}$$

A

$$\begin{bmatrix} 0&-&+&+&-&-&+&+&+&+ \\ -&0&-&+&+&+&-&+&+&+ \\ +&-&0&-&+&+&+&-&+&+ \\ +&+&-&0&-&+&+&+&-&+ \\ -&+&+&-&0&+&+&+&+&- \\ -&+&+&+&+&0&+&-&-&+ \\ +&-&+&+&+&+&0&+&-&- \\ +&+&-&+&+&-&+&0&+&- \\ +&+&+&-&+&-&-&+&0&+ \\ +&+&+&+&-&+&-&-&+&0 \end{bmatrix}$$

Σ

SRGs and ETFs

Notation: For $M \in \mathbb{Z}^{n\times n}$, $\overline{M} \in \mathbb{F}_p^{n\times n}$ is its image mod p

Theorem

Given an SRG with $f \neq g$, define $G := \Sigma + (2r+1)I$ and $d := \operatorname{rank}_p \overline{G}$. If p divides $v - 4k + 2\lambda + 2\mu$, then $\overline{G}$ is the Gram matrix of a $(2r+1, 1, 2r-2s)$-ETF with size $d \times v$ in an orthogonal geometry on $\mathbb{F}_p^d$. Furthermore:

(a) *If $r \not\equiv_p s$ and $v - 2k + 2r \not\equiv_p 0$, then $d = g + 1$.*

(b) *If $r \not\equiv_p s$ and $v - 2k + 2r \equiv_p 0$, then $d = g$.*

(c) *If $r \equiv_p s$, then $d \leq \min\{f+1, g+1\}$.*

(d) *Let p^m be the largest power of p that divides v. If p^{m+1} divides $v - 2k + 2r$, then $d \leq g$.*

More SRGs and ETFs

Theorem

Given an SRG with $f \neq g$, define $d := \operatorname{rank}_p \overline{\Sigma + (2r+1)I}$, $n := v + 1$, and

$$G := \begin{bmatrix} 2r+1 & \mathbf{1}_v^\top \\ \mathbf{1}_v & \Sigma + (2r+1)I \end{bmatrix} \in \mathbb{Z}^{n \times n},$$

If p divides both $k - 2\mu$ and $v - 3k + 2\lambda + 1$, then $\overline{G} \in \mathbb{F}_p^{n \times n}$ is the Gram matrix of a $(2r+1, 1, 2r-2s)$-ETF with size $d \times n$ in an orthogonal geometry on $\mathbb{F}_p^d$. Furthermore:

(a) *If $r \not\equiv_p s$, then $d = g + 1$.*

(b) *If $r \equiv_p s$, then $d \leq \min\{f+1, g+1\}$.*

SRGs from ETFs

Corollary

A (v, k, λ, μ)-SRG comes from a finite field ETF unless **both** *of*

$$|v - 4k + 2\lambda + 2\mu| \qquad \textit{and} \qquad \gcd(k - 2\mu, v - 3k + 2\lambda + 1)$$

are powers of 2.

- Brouwer's table of feasible SRG parameters: 211 known complementary pairs of parameters with $v \leq 1300$
- All but 9 pairs (95%) come from finite field ETFs:

v	k	λ	μ
21	10	3	6
40	12	2	4
57	24	11	9

v	k	λ	μ
70	27	12	9
112	30	2	10
120	42	8	18

v	k	λ	μ
220	84	38	28
280	117	44	52
512	196	60	84

$(a, 1, c)$-ETFs from SRGs

p	d	n	a	c
3	**4**	**10**	**0**	**0**
3	9	25	0	1
3	9	37	0	1
3	10	37	0	0
3	**10**	**55**	**0**	**0**
3	12	36	1	0
3	12	49	0	1
3	12	67	0	1
3	**13**	**91**	**0**	**0**
3	14	36	1	0
3	14	45	1	0
3	15	64	0	1
3	15	106	0	1
3	16	82	0	0
3	**16**	**136**	**0**	**0**
3	18	81	1	0
3	18	100	0	1
3	18	154	0	1
3	19	49	1	1
3	19	65	1	2
3	19	81	1	0
3	19	105	1	0
3	**19**	**190**	**0**	**0**
3	20	46	0	0
3	20	57	1	0
3	21	81	1	0
3	21	121	0	1
3	21	162	1	0
3	21	211	0	1
3	22	50	1	2
3	22	65	1	2
3	22	77	1	2

p	d	n	a	c
3	22	100	1	1
3	22	145	0	0
3	22	243	1	0
3	22	253	0	0
3	**22**	**253**	**1**	**1**
3	22	276	1	0
3	24	117	1	0
3	24	169	0	1
3	24	277	0	1
3	25	101	1	2
3	**25**	**325**	**0**	**0**
5	9	26	0	0
5	10	45	2	4
5	11	56	2	2
5	12	26	0	0
5	**12**	**78**	**2**	**3**
5	13	49	2	1
5	15	65	2	4
5	15	105	2	4
5	16	81	2	2
5	16	121	2	2
5	**17**	**153**	**2**	**3**
5	19	126	0	0
5	20	56	0	2
5	20	81	0	2
5	20	190	2	4
5	21	51	0	0
5	21	176	0	0
5	21	211	2	2
5	**22**	**253**	**2**	**3**
5	23	101	0	0

p	d	n	a	c
5	23	144	2	1
5	24	101	0	0
5	25	81	0	2
5	25	101	0	2
5	25	170	2	4
5	25	300	2	4
7	12	66	3	6
7	13	79	3	1
7	**14**	**105**	**3**	**5**
7	16	50	0	0
7	17	81	3	4
7	19	101	3	6
7	19	171	3	6
7	20	121	3	1
7	20	191	3	1
7	21	77	2	5
7	21	231	3	5
7	24	50	0	0
7	24	100	2	6
11	16	120	3	6
11	17	137	3	8
11	**18**	**171**	**3**	**1**
11	25	169	3	4
13	18	153	3	6
13	19	172	3	8
13	**20**	**210**	**3**	**12**
17	22	231	3	6
17	23	254	3	8
17	**24**	**300**	**3**	**12**
19	24	276	3	6
19	25	301	3	8

Necessary conditions for SRGs

If the SRG exists, then so does the $(a, 1, c)$-ETF

v	k	λ	p	d	n	a	c
69	48	32	13	24	69	5	3
85	54	33	7	35	85	0	4
85	70	57	5	35	86	0	4
85	70	57	13	35	85	5	1
88	27	6	5	32	88	2	3
99	**14**	**1**	**5**	**45**	**100**	**2**	**4**
100	33	8	3	34	101	1	2
111	30	5	19	36	111	7	1
112	36	10	3	48	112	0	2
115	18	1	3	46	115	1	1
115	18	1	17	45	115	7	16
120	34	8	7	52	121	2	6
120	84	58	3	57	121	0	2
121	36	7	3	37	121	1	1
121	36	7	5	36	121	2	2
121	48	17	3	48	121	0	1
133	32	6	3	57	133	0	2
133	32	6	11	56	133	9	9
133	88	57	3	57	133	0	1
133	88	57	5	56	133	4	2
133	108	87	5	57	133	2	3
133	108	87	11	56	133	7	7
136	105	80	3	52	137	1	2
136	105	80	11	52	136	7	9
162	21	0	5	57	163	2	3
162	21	0	7	56	162	0	4
162	92	46	7	24	163	5	1
162	112	76	13	64	162	9	11
162	138	117	7	70	162	0	4
162	138	117	17	70	163	7	1
169	42	5	3	43	170	1	2
169	42	5	5	43	169	2	1
169	42	5	7	42	169	0	5
169	56	15	3	57	169	0	2
169	56	15	5	56	169	4	1
169	70	27	3	70	169	2	2
175	108	63	13	43	175	7	4
176	25	0	3	55	176	1	2

v	k	λ	p	d	n	a	c
176	25	0	7	56	176	0	6
176	25	0	17	56	177	7	3
176	70	24	5	56	177	4	3
183	52	11	5	61	184	4	1
183	52	11	29	60	183	9	26
189	128	82	11	29	189	5	3
189	140	103	5	91	189	1	4
190	144	108	5	76	191	4	4
196	39	2	3	49	197	1	2
196	39	2	7	49	197	0	5
196	39	2	31	48	196	7	26
196	45	4	3	46	196	1	1
196	75	26	3	76	197	2	1
196	114	59	5	25	196	0	2
204	28	2	13	84	204	9	7
204	140	94	3	69	205	0	1
205	68	15	3	40	205	1	1
205	68	15	5	40	205	2	4
208	45	8	5	91	209	1	4
209	156	115	5	76	209	4	1
209	156	115	7	77	210	2	5
209	156	115	11	77	209	9	4
210	33	0	5	55	210	2	4
210	33	0	7	56	211	0	3
210	76	26	5	96	211	3	3
210	132	82	3	100	211	1	1
216	129	72	13	44	217	7	10
216	172	136	5	86	216	4	4
217	128	72	3	63	217	0	2
220	72	22	3	100	220	1	1
225	48	3	7	49	225	0	2
225	64	13	7	64	225	2	2
225	128	64	5	25	226	0	3
231	160	110	5	111	231	3	3
232	33	2	3	87	232	0	1
232	33	2	19	88	232	9	3
232	33	2	23	88	233	9	22
232	63	14	3	88	233	2	1

Bounds on equiangular lines

Problem (Relative bound)

Given an orthogonal geometry on $\mathbb{F}_q^d$ and $a, b \in \mathbb{F}_q$, find an upper bound for the size of an (a, b)-equiangular system.

- Efficient bound may disprove SRGs
- Is there a $(2, 1)$-equiangular system with 100 vectors in an orthogonal geometry on $\mathbb{F}_5^{45}$?
- If not, then Conway's 99-graph DNE

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Theorem (Gerzon; Greaves, JI, Jasper, Mixon)

If $a^2 \neq b$ and there is an (a, b)-equiangular system Φ of n vectors in an orthogonal geometry on $\mathbb{F}_q^d$, then $n \leq \binom{d+1}{2}$. If equality holds, then Φ is an ETF.

- $a^2 \neq b$ avoids trivialities like an entire isotropic subspace

Attaining Gerzon's bound

Theorem (Greaves, JI, Jasper, Mixon)

For every integer $d > 1$ and every odd prime p that divides $d - 7$, there exists a $(3, 1, 12)$-ETF of $n = \binom{d+1}{2}$ vectors in an orthogonal geometry on $\mathbb{F}_p^d$. In particular, Gerzon's bound is attained in some d-dimensional orthogonal geometry provided $|d - 7| \neq 2^k$.

- Complement of the triangular graph $T(d+1)$, i.e., Kneser graph $K(d+1, 2)$
- Add some identity to the Seidel matrix
- Take it mod p
- Factor the Gram matrix

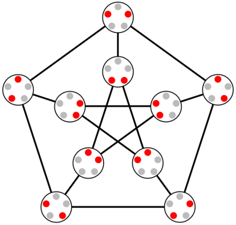

Image from wikimedia.org

Not attaining Gerzon's bound

Recall: We win with Gerzon if $|d-7| \neq 2^k$

Theorem (Greaves, JI, Jasper, Mixon)

For any choice of odd prime power q, there does not exist an ETF of $n = 15 = \binom{5+1}{2}$ vectors in any orthogonal geometry on $\mathbb{F}_q^5$. In particular, Gerzon's bound is not attained in any 5-dimensional finite orthogonal geometry over a field of odd characteristic.

- Real 5×15 ETF DNE
- Overlap with real case: $p \leq 25$
- Field stuff: $q = p \in \{3, 5, 7, 19\}$
- Witt extension theorem: Computationally efficient way to bound a clique number in graph of vectors with the right angle
- $p = 5$ is a weirdo, comes down to rank condition

Outline

Context: Real and complex equiangular lines

Equiangular lines in orthogonal geometry

Equiangular lines in unitary geometry

Unitary geometry

Throughout this section, q is a prime power (possibly even)

"Complex conjugation" on $\mathbb{F}_{q^2}$: $x \mapsto x^q$

- Order-2 field automorphism
- Fixes $\mathbb{F}_q \le \mathbb{F}_{q^2}$

Unitary geometry

We give $\mathbb{F}_{q^2}^d$ the Hermitian form $\langle x, y \rangle := \sum_{i \in [d]} x_i^q y_i =: x^* y$.

For every $x, y \in \mathbb{F}_{q^2}^d$:

- $\langle x, y \rangle = \langle y, x \rangle^q$
- $\langle x, \cdot \rangle \colon \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2}^d$ is linear
- if $x \neq 0$, then $\langle x, \cdot \rangle \colon \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2}^d$ is not the zero mapping

This is the unique Hermitian form (up to isometric isomorphism).

Equiangular systems and tight frames

Definition

For $a, b, c \in \mathbb{F}_q$, a sequence $\varphi_1, \ldots, \varphi_n \in \mathbb{F}_{q^2}^d$ is:

- a **frame** if it spans $\mathbb{F}_{q^2}^d$;
- a c-**tight frame** if it spans $\mathbb{F}_{q^2}^d$ and moreover
  $$\sum_{j\in[n]} \langle \varphi_j, x\rangle \varphi_j = cx$$
  for every $x \in \mathbb{F}_{q^2}^d$;
- an (a, b)-**equiangular system** if:
  (i) $\langle \varphi_i, \varphi_i \rangle = a$ for every $i \in [n]$, and
  (ii) $\langle \varphi_i, \varphi_j \rangle^{q+1} = b$ for every $i \neq j$ in $[n]$;
- an (a, b, c)-**equiangular tight frame** (ETF) if it is an (a, b)-equiangular system and a c-tight frame.

Some examples

Example

Any ETF in an orthogonal geometry on $\mathbb{F}_q^d$ gives one of the same size in unitary geometry on $\mathbb{F}_{q^2}^d$.

Example

$q = 2$, $\zeta \in \mathbb{F}_{2^2}^\times$ is primitive.
There is a 6×27 ETF with parameters $(a, b, c) = (0, 1, 1)$:

$$\Phi = \begin{bmatrix} 1 & \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta \\ \zeta & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \zeta & \zeta & \zeta^2 & \zeta^2 & 0 & 0 & 0 & 0 & 1 & \zeta & \zeta^2 \\ 0 & 0 & 1 & 1 & 1 & \zeta & \zeta^2 & \zeta^2 & 1 & 1 & \zeta & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^2 & 0 & 0 & \zeta & \zeta & \zeta & \zeta & \zeta \\ 0 & 0 & 1 & \zeta & \zeta & 0 & 1 & \zeta^2 & \zeta & \zeta^2 & 0 & 1 & 1 & \zeta & 0 & \zeta & 0 & \zeta & 0 & \zeta & 1 & \zeta & 0 & \zeta^2 & \zeta & \zeta & \zeta \\ 0 & 0 & \zeta & \zeta & 1 & 0 & \zeta^2 & 1 & \zeta^2 & \zeta & 0 & \zeta & 1 & 1 & \zeta & 0 & \zeta & 0 & \zeta & 0 & \zeta & 1 & \zeta^2 & 0 & \zeta & \zeta & \zeta \end{bmatrix}$$

Its automorphism group is doubly transitive. (No doubly transitive complex ETF of this size exists.)

Projecting complex ETFs

Theorem (Greaves, JI, Jasper, Mixon)

Suppose there is a $d \times n$ complex ETF.

(a) *There is a $d \times n$ complex ETF with algebraic entries.*

(b) *For infinitely many pairwise coprime q, there is a $d \times n$ ETF in a unitary geometry on $\mathbb{F}_{q^2}^d$.*

- (a) is an application of Tarski–Seidenberg
- (b) is an application of Frobenius density theorem

Problem

Does the converse of (b) hold?

Gerzon's bound in unitary geometry

Theorem

If $a^2 \neq b$ and there is an (a, b)-equiangular system Φ of n vectors in a unitary geometry on $\mathbb{F}_{q^2}^d$, then $n \leq d^2$. If equality holds, then Φ is an ETF.

Problem

For which (q, d) is Gerzon's bound saturated?

- Zauner: For every d, there exists q
- Harder than Zauner to solve completely
- Easier to make progress

Time-frequency shifts

Assume $d_1, \ldots, d_m$ **all divide** $q+1$**.** Let $G = \prod_{k=1}^m \mathbb{Z}/d_k\mathbb{Z}$.

$\mathbb{F}_{q^2}^G \cong \mathbb{F}_{q^2}^{|G|}$ consists of functions $\varphi\colon G \to \mathbb{F}_{q^2}$.

For each k, fix a generator $\omega_k \in \mathbb{F}_{q^2}^\times$ for the subgroup of size d_k

Definition

For $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m) \in G$ and $\varphi \in \mathbb{F}_{q^2}^G$,

$$(T_y\varphi)(x) := \varphi(x-y) \qquad \text{and} \qquad (M_y\varphi)(x) := \prod_{k=1}^m \omega_k^{x_k y_k} \cdot \varphi(x).$$

Then $T_y, M_y \colon \mathbb{F}_{q^2}^G \to \mathbb{F}_{q^2}^G$ are isometric isomorphisms.

Proposition

For any nonzero $\varphi \in \mathbb{F}_{q^2}^G$, $\{T_x M_y \varphi\}_{x,y\in G}$ *is a tight frame for* $\mathbb{F}_{q^2}^G$.

Attaining Gerzon's bound

Theorem (Greaves, JI, Jasper, Mixon)

Take $q = 3$, m odd, and $d_k = 2$ for $1 \le k \le m$, so $G = (\mathbb{Z}/2\mathbb{Z})^m$. Let $\zeta \in \mathbb{F}_{3^2}$ be primitive, and define $\varphi \in \mathbb{F}_{3^2}^G$ by

$$\varphi(x) = \begin{cases} -1 - \zeta^2, & \text{if } x = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\Phi = \{T_x M_y \varphi\}_{x,y \in G}$ is a $(0,1,0)$-ETF of size $2^m \times 2^{2m}$. In particular, Gerzon's bound is attained in a unitary geometry on $\mathbb{F}_{3^2}^d$ whenever $d = 2^{2k+1}$.

- "TF" is free
- "E" is a direct calculation

Limited connections with complex case

Recall: Gerzon's bound is attained in a unitary geometry on $\mathbb{F}_{3^2}^d$ whenever $d = 2^{2k+1}$

- $d = 2, 8$ lift to complex ETFs that attain Gerzon's bound
  - $d = 2$: Tetrahedron in Bloch sphere
  - $d = 8$: Hoggar's 64 lines in $\mathbb{C}^8$
- $d = 32$ does not appear to lift to a complex ETF
- Godsil and Roy: Time-frequency shifts over $\mathbb{Z}_2^m$ generate an ETF only if $m \in \{1, 3\}$.

Godsil and Roy, European J. Combin., 2009

Some more problems

Problem

Generalize (doubly transitive) complex 3×9 to an infinite family over a finite field.

Problem

Is there a combinatorial description of ETFs in finite unitary geometry?

Problem

Find necessary conditions (e.g. integrality constraints) for ETF existence in unitary geometry.

Questions?

Frames over finite fields: Basic theory and equiangular lines in unitary geometry
G.R.W. Greaves, J.W. Iverson, J. Jasper, D.G. Mixon
arXiv:2012.12977

Frames over finite fields: Equiangular lines in orthogonal geometry
G.R.W. Greaves, J.W. Iverson, J. Jasper, D.G. Mixon
arXiv:2012.13642

Image from `sideshowtoy.com`